

AN ALGEBRAIC MODEL FOR RATIONAL TORUS-EQUIVARIANT SPECTRA

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ABSTRACT. We provide a universal de Rham model for rational G -equivariant cohomology theories for an arbitrary torus G . More precisely, we show that the representing category, of rational G -spectra, is Quillen equivalent to an explicit small and practical algebraic model.

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Part 1. Overview and context

1. INTRODUCTION

1.A. Preamble. Cohomology theories are contravariant homotopy functors on topological spaces satisfying the Eilenberg-Steenrod axioms (except for the dimension axiom), and any cohomology theory $E^*(\cdot)$ is represented by a homotopy theoretic spectrum E in the sense that $E^*(X) = [X, E]^*$. Accordingly, the category of spectra gives an embodiment of the category of cohomology theories in which one can do homotopy theory. The complexity of the homotopy theory of spectra is visible even in the homotopy endomorphisms of the unit object: this is the ring of stable homotopy groups of spheres, and this is so intricate that we cannot expect a complete analysis of the category of spectra in general. However, most of the complication comes from \mathbb{Z} -torsion so we can simplify things by rationalizing. The resulting category of rational spectra represents cohomology theories with values in rational vector spaces. The simplicity of this rationalized category is apparent by Serre's theorem: the rationalization of the stable homotopy groups of spheres simply consists of \mathbb{Q} in degree 0, and it is a small step to see that there is nothing more to the topology of rational cohomology theories than their graded rational vector space of coefficients. On the other hand, de Rham cohomology shows that a large amount of useful geometry remains even when we rationalize. Accordingly, the study of rational cohomology theories and rational spectra is both accessible and useful.

These facts are well-known, and it is natural to ask what happens when we consider spaces with an action of a compact Lie group G . Once again, a G -equivariant cohomology theory is a contravariant homotopy functor on G -spaces satisfying suitable conditions, and each such G -equivariant cohomology theory is represented by a G -spectrum [33]. In the equivariant case, when we rationalize a G -spectrum, considerably more structure remains than in the non-equivariant case. It is natural to expect rational representation theory to play a role in understanding rational equivariant cohomology theories, and when G is finite this is the only ingredient. However in general, the other significant piece of structure is exemplified by the Localization Theorem: for a torus G this states that for finite complexes there is no difference between the Borel cohomology of a G -space and its G -fixed points once the Euler classes are inverted. These ingredients are used to build the algebraic model [18] for rational G -spectra described in Section 2 below.

The archetype for giving an algebraic model for the homotopy theory of topological origin is Quillen's analysis of simply connected rational spaces [39]. To prove the result he introduced the axiomatic framework of model categories which underly the homotopy category, and the notion of a Quillen equivalence between model categories preserving the homotopy theories. The use of these ideas is now widespread, and we refer to [27] and [26] for details.

Our main result is a Quillen equivalence between the category of rational G -spectra for a torus G and an explicit and calculable algebraic model. In the course of our proof, we introduce a number of techniques of broader interest, in equivariant homotopy theory and in the theory of model categories. In the rest of the introduction, we give a little history, and then describe our results, methods and conventions.

1.B. Equivariant cohomology theories. Non-equivariantly, rational stable homotopy theory is very simple: the homotopy category of rational spectra is equivalent to the category of graded rational vector spaces, and all cohomology theories are ordinary in the sense that they are naturally equivalent to ordinary cohomology with coefficients in a graded vector space. The first author has conjectured [17] that for each compact Lie group G , there is an abelian category $\mathcal{A}(G)$, so that the homotopy category of rational G -spectra is equivalent to the homotopy category of differential graded objects of $\mathcal{A}(G)$:

$$\mathrm{Ho}(G\text{-spectra}/\mathbb{Q}) \simeq \mathrm{Ho}(DG - \mathcal{A}(G)).$$

In general terms, the objects of $\mathcal{A}(G)$ are sheaves of graded modules with additional structure over the space of closed subgroups of G , with the fibre over H giving information about the H -fixed points. The conjecture describes various properties of $\mathcal{A}(G)$, and in particular asserts that its injective dimension is equal to the rank of G . According to the conjecture one may therefore expect to make complete calculations in rational equivariant stable homotopy theory, and one can classify cohomology theories. Indeed, one can construct a cohomology theory by writing down a differential graded object in $\mathcal{A}(G)$: this is how $SO(2)$ -equivariant elliptic cohomology was constructed in [20], and it is hoped to construct cohomology theories associated to generic curves of higher genus in a similar way using the results of this paper.

The conjecture is elementary for finite groups, where $\mathcal{A}(G) = \prod_{(H)} \mathbb{Q}W_G(H)\text{-mod}$ [22], where the product is over conjugacy classes of subgroups H and $W_G(H) = N_G(H)/H$. This means that any cohomology theory is again ordinary in the sense that it is a sum over conjugacy classes (H) of ordinary cohomology of the H -fixed points with coefficients in a graded $\mathbb{Q}W_G(H)$ -module. The conjecture has been proved for the rank 1 groups $G = SO(2), O(2), SO(3)$ in [15, 14, 16], where $\mathcal{A}(G)$ is more complicated. It is natural to go on to conjecture that the equivalence comes from a Quillen equivalence

$$G\text{-spectra}/\mathbb{Q} \simeq DG - \mathcal{A}(G),$$

for suitable model structures. The second author has established this for $G = SO(2)$ [45], Barnes [3, 4] has shown how to deduce it for $G = O(2)$ from a suitable proof for $G = SO(2)$ (such as the one we use here). This earlier work relied on the particular simplicity of the case $r = 1$, and there is no prospect of extending the methods of [15] or [45] to higher rank. Indeed, even if one only wants an equivalence of triangulated categories, it appears essential to establish the Quillen equivalence when $r \geq 2$.

1.C. The classification theorem. The present paper completes the programme begun in [18, 19]. The purpose of the series is to provide a small and calculable algebraic model for rational G -equivariant cohomology theories for a torus G of rank $r \geq 0$. Such cohomology theories are represented by rational G -spectra, and in this paper we show that the category of rational G -spectra is Quillen equivalent to the small and concrete abelian category $\mathcal{A}(G)$ introduced in [18] (its definition and properties are summarized in Section 2). The category $\mathcal{A}(G)$ is designed as a natural target of a homology theory

$$\pi_*^{\mathcal{A}} : G\text{-spectra} \longrightarrow \mathcal{A}(G);$$

the idea is that $\mathcal{A}(G)$ is a category of sheaves of modules, with the stalk over a closed subgroup H being the Borel cohomology of the geometric H -fixed point set with suitable coefficients. A main theorem of [18] shows that $\mathcal{A}(G)$ is of finite injective dimension (shown in [19] to be r).

The main theorem of the present paper and the culmination of the series is as follows. Model structures will be described in Sections 3 and 16 below.

Theorem 1.1. *For any torus G , there is a Quillen equivalence*

$$G\text{-spectra}/\mathbb{Q} \simeq_Q DG - \mathcal{A}(G)$$

of model categories. In particular their homotopy categories are equivalent

$$Ho(G\text{-spectra}/\mathbb{Q}) \simeq Ho(DG - \mathcal{A}(G))$$

as triangulated categories.

Remark 1.2. The functors involved in these Quillen equivalences are monoidal, but their interaction with the model structures is not straightforward. For this reason, we plan to describe the extension of this result to Quillen equivalences on the associated categories of monoids elsewhere.

Because of the nature of the theorem, it is easy to impose restrictions on the isotropy groups occurring in topology and algebra, and one may deduce versions of this theorem for categories of spectra with restricted isotropy groups. For example we recover a special case of the result of [23], which states that if G is any connected compact Lie group there is a Quillen equivalence

$$\text{free-}G\text{-spectra}/\mathbb{Q} \simeq_Q \text{DG-torsion-}H^*(BG)\text{-modules},$$

with a quite different proof. The methods of the present paper are used to extend the result on free G -spectra to disconnected groups G in [24].

1.D. Applications. Beyond the obvious structural insight, the type of applications we anticipate may be seen from those already given for the circle group \mathbb{T} (i.e., the case $r = 1$). For example [15] gives a classification of rational \mathbb{T} -equivariant cohomology theories, a precise formulation and proof of the rational \mathbb{T} -equivariant Segal conjecture, and an algebraic analysis of existing theories, such as K -theory and the construction of topological cyclic from topological Hochschild homology. More significant is the construction in [20] of a rational equivariant cohomology theory associated to an elliptic curve C over a \mathbb{Q} -algebra, and the identification of a part of \mathbb{T} -equivariant stable homotopy theory modelled on the derived category of sheaves over C . The philosophy in which equivariant cohomology theories correspond to algebraic groups is expounded in [21], and there are encouraging signs suggesting that one may use the model described in the present paper to construct torus-equivariant cohomology theories associated to generic complex curves of higher genus.

1.E. Outline of strategy. The proof is made possible by the apparatus of model categories and the existence of a good symmetric monoidal category of spectra, allowing us to talk about commutative ring spectra and modules over them. There are two other particular ingredients. The second author's results [46] gives Quillen equivalences between algebras over the Eilenberg-MacLane spectrum $H\mathbb{Q}$ and differential graded \mathbb{Q} -algebras, and between the module categories of corresponding algebras; this allows us to pass from topology to algebra. Finally, the first author's [18] defining the algebraic category $\mathcal{A}(G)$ provides an algebraic model and the Adams spectral sequence based on it gives a means for calculation in the homotopy category.

In outline, what we have to achieve is to move from the category of rational G -spectra to the category of DG objects of the abelian category $\mathcal{A}(G)$. There are five main stages to this.

Isotropy separation: (Sections 4 to 10) We replace the category of G -spectra, which is the category of modules over the sphere spectrum, by a category of diagrams of modules over commutative equivariant ring spectra. Each of these module categories captures information about subgroups with a specified identity component and the diagram shows how to reassemble this local information into a global spectrum.

Removal of equivariance: (Sections 11 to 13) At each point in the diagram, we replace the category of modules over commutative ring G -spectra by a category of modules over a commutative non-equivariant ring spectrum by passage to fixed points.

Transition to algebra: (Section 14) At each point in the diagram, we apply the second author's machinery to replace all the commutative ring spectra in the diagram by commutative DGAs, and the category of module spectra by the corresponding category of DG modules over the DGAs.

Rigidity: (Section 15) The commutative DGAs and the construction of the diagram is such that it is determined by the homology. Accordingly the diagram of commutative DGAs may be replaced by a diagram of commutative algebras.

Simplification: (Sections 16 and 17) At each stage so far, we use cellularization to modify the model structures so that the homotopy category selects the localizing subcategory built from certain specified 'cells'. The final step is to replace this cellularization of the category of DG-modules over the diagram of commutative rings by a much smaller category of modules with special properties, so that no cellularization is necessary; this category turns out to be $\mathcal{A}(G)$.

1.F. **Highlights.** Each of the above steps has some intrinsic interest outside the present project, and we highlight four significant ingredients.

First, we use the Cellularization Principle. The idea is that a Quillen adjunction induces a Quillen equivalence between cellularized model categories; we may choose the collection of cells in either one of the categories, and then use their images in the other (see Proposition A.6). By choosing cells and their images, the cellularization of the adjunction induces a homotopy category level equivalence between the respective localizing subcategories. The hypotheses are mild, and it may appear like a tautology, but it has been useful innumerable times in the present paper and deserves emphasis. It can be directly compared to another extremely powerful formality, that a natural transformation of cohomology theories that is an isomorphism on spheres is an equivalence.

Second, we make extensive use of categories of modules over diagrams of rings, and prove that up to Quillen equivalence and cellularization, we can omit entries in the diagram that can be filled in by extension of scalars or pullback.

Third, the fact that if A is a ring G -spectrum, passage to Lewis-May K -fixed points establishes a close relationship between the category of A -module G -spectra and the category of A^K -module G/K -spectra. More precisely, we consider a Quillen adjunction

$$A \otimes_{A^K} (\cdot) : A^K\text{-mod-}G/K\text{-spectra} \rightleftarrows A\text{-mod-}G\text{-spectra} : (\cdot)^K ,$$

and apply the Cellularization Principle A.6. Other examples of this deserve investigation.

Finally, we note that at the centre of the proof is rigidity: any two model categories with suitable specified homotopy level properities are equivalent. This can be viewed as a recognition principle: we recognize a topological model category as equivalent to an algebraic model because its homotopy category has the required homotopy level properties. We have used only one basic rigidity result: any two commutative DGAs which have the same polynomial cohomology are quasi-isomorphic. This elementary result has far reaching consequences. Our main use of it here is to patch together local rigidity results (each based on polynomial rings) to give a global rigidity result. In [23] we applied it to prove rigidity of Koszul duals. We also need a rigidity result for modules, that by an Adams spectral sequence argument, the standard cells are determined by their homology [18, 12.1].

1.G. Prospects. Based on the ideas of the present paper there is a clear strategy for establishing an algebraic model for rational G -spectra for an arbitrary compact Lie group G . Rational G -spectra would be shown to be given by a G -sheaf of equivariant spectra over the space of closed subgroups, with some additional structure. In a suitable formal context and with checks on the continuity of various constructions, it would be sufficient to argue stalkwise as follows.

First, a G spectrum X corresponds to a G -sheaf $\widetilde{M}(X)$ of equivariant module spectra over a G -sheaf \widetilde{R}_{top} of equivariant ring spectra. Choosing a particular closed subgroup H , we note that it is fixed by the normalizer $N_G(H)$, and the fibre $\widetilde{R}_{top}(H)$ over H is an $N_G(H)$ -ring spectrum, with $\widetilde{M}(X)_H$ a module $N_G(H)$ -spectrum over it. Passing to fixed points under H we obtain a $W_G(H)$ -spectrum. Passing to fixed points under the identity component $W_G(H)_1$, we obtain a ring $\pi_0(W_G(H))$ -ring spectrum $R_{top}(H)$, and the corresponding fixed point spectrum $M(X)_H$ of $\widetilde{M}(X)_H$ is a module over it. Note that now the only equivariance remaining is for the *finite* component group $\pi_0(W_G(H))$, and since we are working over the rationals, it is easy to see this structure is equivalent to a suitable piece of rational representation theory.

By the second author's theorem, this category of modules over the $\pi_0(W_G(H))$ -ring spectrum $R_{top}(H)$ is equivalent to a category of sheaves of modules over a DGA $R_t(H)$ with an action of $\pi_0(W_G(H))$. By intrinsic formality of the cohomology of $R_t(H)$ this is equivalent to a category of modules over the polynomial ring $H^*(BW_G(H)_1)$ with a twisted action of $\pi_0(W_G(H))$.

Reassembling, we find that we now have an equivariant module over a sheaf R_a of rings, whose fibre over H is $H^*(BW_G(H)_1)$, and the fibre has an action of $\pi_0(W_G(H))$. Finally the cellularized algebraic category would be shown to be equivalent to a subcategory of the category of equivariant modules over the sheaf of rings.

Evidently a substantial technical apparatus must be developed to implement this strategy. The resulting model would then give explicit calculations by choosing a suitable cover of the space of subgroups over each set of which the category of modules is well understood. By contrast, the method of the present paper takes advantage of the simple subgroup structure of tori to combine the two steps, and to avoid the technical obstacles of dealing with equivariant spectra over a G -space.

1.H. Relationship to other results. We should comment on the relationship between the strategy implemented here and that used for free spectra in [23]. Both strategies start with a category of G -spectra and end with a purely algebraic category, and the connection in

both relies on finding an intermediate category which is visibly rigid in the sense that it is determined by its homotopy category (the archetype of this is the category of modules over a commutative DGA with polynomial cohomology).

The difference comes in the route taken. Roughly speaking, the strategy in [23] is to move to non-equivariant spectra as soon as possible, whereas that adopted here is to keep working in the ambient category of G -spectra for as long as possible.

The advantage of the strategy of [23] is that it is close to commutative algebra, and should be adaptable to proving uniqueness of other algebraic categories. However, it is hard to retain control of the monoidal structure, and adapting the method to deal with many isotropy groups makes the formal framework very complicated. This was our original approach to the result for tori.

The present method appears to have several advantages. It uses fewer steps, and the monoidal structures are visible throughout. Furthermore, it reflects traditional approaches to the homotopy theory of G -spaces in that it displays the category of G -spectra as built from categories of spectra with restricted isotropy group using Borel cohomology.

1.1. Conventions. Certain conventions are in force throughout the paper and the series. The most important is that *everything is rational*: henceforth all spectra and homology theories are rationalized without comment. For example, the category of rational G -spectra will now be denoted ‘ G -spectra’. Whenever possible we work in the derived category; for example, most equivalences are verified at this level. We also use the standard conventions that ‘DG’ abbreviates ‘differential graded’ and that ‘subgroup’ means ‘closed subgroup’. We attempt to let inclusion of subgroups follow the alphabet, so that $G \supseteq H \supseteq K \supseteq L$. We focus on homological (lower) degrees, with differentials reducing degrees; for clarity, cohomological (upper) degrees are called *codegrees* and may be converted to degrees by negation in the usual way. Finally, we write $H^*(X)$ for the unreduced cohomology of a space X with rational coefficients.

We have adopted a number of more specific conventions in our choice of notation, and it may help the reader to be alerted to them.

- There are several cases where we need to talk about ring G -spectra \tilde{R} and their fixed points $R = (\tilde{R})^G$. The equivariant form is indicated by a tilde on the non-equivariant one.
- We need to discuss rings in various categories of spectra, and then modules over them. Since it often needs to be made explicit, we write, for example, R -module- G -spectra for the category of R -modules in the category of G -spectra.
- We will not usually make explicit the universe over which our spectra are indexed. The default is that a category of G -spectra will be indexed over a complete G -universe, and we only highlight the universe in other cases or when it needs emphasis.
- The purpose of this paper is to give an algebraic model of a topological phenomenon. Accordingly, characters arise in various worlds, and it is useful to know they play corresponding roles. We sometimes point this out by use of subscripts. For example R_a (with ‘ a ’ for ‘algebra’) might be a (conventional, graded) ring, R_{top} its counterpart in spectra, \tilde{R}_{top} its counterpart in G -spectra, and R_t a large and poorly understood DGA of topological origin.

- We often have to discuss diagrams of rings and diagrams of modules over them, but we will usually say that R is a diagram of rings and M is an R -module (leaving the fact that M is also a diagram to be deduced from the context).

1.J. Organization of the paper. Sections 2 and 3 give background, recalling the definition of the algebraic model $\mathcal{A}(G)$ and basic facts about the ring G -spectra that we use.

Section 4 explains the key ingredients, and gives a complete outline in the simple case of the circle.

Most of the rest of the paper is devoted to establishing the following sequence of Quillen equivalences, several of which are themselves zig-zags of simple Quillen equivalences. The cellularizations are all with respect to the images of the cells G/H_+ for closed subgroups H , and the superscripts refer to two diagrams $\mathbf{LI}(G) \subset \mathbf{LCI}(G)$ based on localization, completion and inflation.

$$\begin{aligned}
G\text{-spectra} &\stackrel{(1)}{\simeq} \text{cell-}\tilde{R}_{top}^{\mathbf{LCI}(G)}\text{-mod-}G\text{-spectra} \stackrel{(2)}{\simeq} \text{cell-}\tilde{R}_{top}^{\mathbf{LI}(G)}\text{-mod-}G\text{-spectra} \\
&\stackrel{(3)}{\simeq} \text{cell-}R_{top}^{\mathbf{LI}(G)}\text{-mod-spectra} \stackrel{(4)}{\simeq} \text{cell-}R_t^{\mathbf{LI}(G)}\text{-mod} \\
&\stackrel{(5)}{\simeq} \text{cell-}R_a^{\mathbf{LI}(G)}\text{-mod} \stackrel{(6)}{\simeq} \text{qce-}R_a^{\mathbf{LI}(G)}\text{-mod} \stackrel{(7)}{=} \mathcal{A}(G)
\end{aligned}$$

Sections 5 to 7 introduce the formalism for discussing modules over diagrams of rings; the particular diagrams $\mathbf{LI}(G)$, $\mathbf{LC}(G)$ and $\mathbf{LCI}(G)$ that we use are based on localization, completion and inflation. By way of example, Section 7 explains Equality (7) stating that $\mathcal{A}(G)$ can be viewed as a category of modules over $\mathbf{LI}(G)$. Section 6 describes a general method for establishing that a restriction of diagrams gives a Quillen equivalence. The particular diagrams \tilde{R}_{top} (of equivariant spectra) and R_{top} (of non-equivariant spectra) are introduced in Section 9.

Sections 8 to 10 use the formalism just introduced to establish Equivalences (1) and (2), showing that rational G -spectra are equivalent to a category of module spectra over an $\mathbf{LI}(G)$ -diagram \tilde{R}_{top} of equivariant ring spectra. This completes the isotropy separation step of the proof.

Until this point, all arguments and calculations are within the category of G -spectra. The remaining steps change ambient categories. We not only need to recognize the categories of modules, but we also need to recognize the cells we use to cellularize them. The fact that the natural cells G/H_+ are characterized by their homology ([18, 12.1]) means that we do not need to comment further on the cells.

Sections 11 to 13 take the next step, establishing Equivalence (3), showing that passage to fixed points gives an equivalence between the $\mathbf{LI}(G)$ -diagram of equivariant \tilde{R}_{top} -module spectra and the $\mathbf{LI}(G)$ -diagram of modules over the split $\mathbf{LI}(G)$ -diagram R_{top} of non-equivariant spectra (the notion of a split diagram is described in Section 13). This completes the removal of equivariance step.

In Section 14, it is explained that the second author's work establishes Equivalence (4) showing that the category of module spectra over this split $\mathbf{LI}(G)$ -diagram R_{top} of ring spectra is equivalent to the category of modules over the split $\mathbf{LI}(G)$ -diagram of DGAs R_t . It is then quite straightforward to establish Equivalence (5), showing in Section 15 that the split $\mathbf{LI}(G)$ -diagram $R_a = H_*(R_t)$ is intrinsically formal, so that the category of modules

over R_t and R_a are equivalent. Finally Sections 16 and 17 establish Equivalence (6), showing that the cellularization is equivalent to the particular category $\mathcal{A}(G)$ of R_a -modules.

Appendix A gives a brief account of the Cellularization Principle, Appendix B shows that the category of modules over a diagram of rings admits a diagram-injective model structure and Appendix C develops a doubly injective (i.e., in both the diagrammatic and the algebraic senses) model structure on the category of modules over a diagram of DGAs.

2. THE ALGEBRAIC MODEL

In this section we briefly recall relevant results from [18] which constructs an abelian category $\mathcal{A}(G)$ giving an algebraic reflection of the structure of the category of G -spectra and an Adams spectral sequence based on it. The structures from that analysis will be relevant to much of what we do here. Section 7 rephrases the definition in terms of the diagrams of this paper, and in the process makes explicit some further details.

2.A. Definition of the category. First we must construct the category $\mathcal{A}(G)$. For the purposes of this paper we view this as a category of modules over a diagram of rings. Consider a category \mathbf{D} and a diagram of R with shape \mathbf{D} . An R -module is given by a diagram M such that $M(x)$ is an $R(x)$ -module for each object x in \mathbf{D} , and for every morphism $a : x \rightarrow y$ in \mathbf{D} , the map $M(a) : M(x) \rightarrow M(y)$ is a module map over the ring map $R(a) : R(x) \rightarrow R(y)$. That is, we restrict scalars on $M(y)$ over the map $R(a)$ and ask that $M(a)$ is an $R(x)$ -module map. See Subsection 5.A for a more complete discussion.

The shape of the diagram for $\mathcal{A}(G)$ is modeled on the partially ordered set $\mathbf{ConnSub}(G)$ of connected subgroups of G . To start with we consider the single graded ring

$$\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(BG/F),$$

where the product is over the family \mathcal{F} of finite subgroups of G . To specify the value of the ring at a connected subgroup K , we use Euler classes: indeed if V is a representation of G we may define $c(V) \in \mathcal{O}_{\mathcal{F}}$ by specifying its components. In the factor corresponding to the finite subgroup F we take $c(V)(F) := c_{|V^F|}(V^F) \in H^{|V^F|}(BG/F)$ where $c_{|V^F|}(V^F)$ is the classical Euler class of V^H in ordinary rational cohomology.

The diagram of rings $\tilde{\mathcal{O}}_{\mathcal{F}}$ is defined by the following functor on $\mathbf{ConnSub}(G)$

$$\tilde{\mathcal{O}}_{\mathcal{F}}(K) = \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}}$$

where $\mathcal{E}_K = \{c(V) \mid V^K = 0\} \subseteq \mathcal{O}_{\mathcal{F}}$ is the multiplicative set of Euler classes of K -essential representations. Each of the Euler classes is a sum of homogeneous terms which have zero product with each other, and so this localization is again a graded ring. Next we consider the category of modules M over the diagram $\tilde{\mathcal{O}}_{\mathcal{F}}$. Thus the value $M(K)$ is a module over $\mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}}$, and if $L \subseteq K$, the structure map

$$M(L) \rightarrow M(K)$$

is a map of modules over the map

$$\mathcal{E}_L^{-1} \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}}$$

of rings. Note this map of rings is a localization since $V^L = 0$ implies $V^K = 0$ so that $\mathcal{E}_L \supseteq \mathcal{E}_K$.

The category $\mathcal{A}(G)$ is formed from a subcategory of the category of $\tilde{\mathcal{O}}_{\mathcal{F}}$ -modules by adding structure. There are two requirements which we briefly indicate here. We make the necessary extra structure explicit in Section 7. Firstly they must be *quasi-coherent*, in that they are determined by their value at the trivial subgroup 1 by the formula

$$M(K) := \mathcal{E}_K^{-1} M(1).$$

The second condition involves the relation between G and its quotients. Choosing a particular connected subgroup K , we consider the relationship between the group G with the collection \mathcal{F} of its finite subgroups and the quotient group G/K with the collection \mathcal{F}/K of its finite subgroups. For G we have the ring $\mathcal{O}_{\mathcal{F}}$ and for G/K we have the ring

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K})$$

where we have identified finite subgroups of G/K with their inverse images in G , i.e., with subgroups \tilde{K} of G having identity component K . Combining the inflation maps associated to passing to quotients by K for individual groups, there is an inflation map

$$\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}.$$

The second condition is that the object should be *extended*, in the sense that for each connected subgroup K there is a specified isomorphism

$$M(K) = \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

for some $\mathcal{O}_{\mathcal{F}/K}$ -module $\phi^K M$. These identifications should be compatible in the evident way when we have inclusions of connected subgroups.

2.B. Connection with topology. The connection between G -spectra and $\mathcal{A}(G)$ is given by a functor

$$\pi_*^{\mathcal{A}} : G\text{-spectra} \longrightarrow \mathcal{A}(G)$$

with the exactness properties of a homology theory. It is rather easy to write down the value of the functor as a diagram of abelian groups.

Definition 2.1. For a G -spectrum X we define $\pi_*^{\mathcal{A}}(X)$ on K by

$$\pi_*^{\mathcal{A}}(X)(K) = \pi_*^G(DEF_+ \wedge S^{\infty V(K)} \wedge X).$$

Here EF_+ is the universal space for the family \mathcal{F} of finite subgroups with a disjoint basepoint added and $DEF_+ = F(EF_+, S^0)$ is its functional dual (the function G -spectrum of maps from EF_+ to S^0). The G -space $S^{\infty V(K)}$ is defined by

$$S^{\infty V(K)} = \lim_{\rightarrow V \supseteq K} S^V,$$

when $K \subseteq H$, so there is a map $S^{\infty V(K)} \longrightarrow S^{\infty V(H)}$ inducing the map $\pi_*^{\mathcal{A}}(X)(K) \longrightarrow \pi_*^{\mathcal{A}}(X)(H)$. \square

The definition of $\pi_*^{\mathcal{A}}(X)$ shows that quasi-coherence for $\pi_*^{\mathcal{A}}(X)$ is just a matter of understanding Euler classes. The extendedness of $\pi_*^{\mathcal{A}}(X)$ is a little more subtle, and will play a

significant role later. For the present, it follows from a construction of the geometric fixed point functor, and it turns out that we may take

$$\phi^K \pi_*^{\mathcal{A}}(X) = \pi_*^{G/K}(DEF/K_+ \wedge \Phi^K(X)),$$

where Φ^K is the geometric fixed point functor.

To see that $\pi_*^{\mathcal{A}}(X)$ is a module over \mathcal{O} , the key is to understand S^0 .

Theorem 2.2. [18, 1.5] *The image of S^0 in $\mathcal{A}(G)$ is the structure sheaf:*

$$\mathcal{O} = \pi_*^{\mathcal{A}}(S^0).$$

Some additional work confirms that $\pi_*^{\mathcal{A}}$ has the appropriate behaviour.

Corollary 2.3. [18, 1.6] *The functor $\pi_*^{\mathcal{A}}$ takes values in the abelian category $\mathcal{A}(G)$.*

2.C. The Adams spectral sequence. The homology theory $\pi_*^{\mathcal{A}}$ may be used as the basis of an Adams spectral sequence for calculating maps between rational G -spectra. The main theorem of [18] is as follows.

Theorem 2.4. ([18, 9.1]) *For any rational G -spectra X and Y there is a natural Adams spectral sequence*

$$\mathrm{Ext}_{\mathcal{A}(G)}^{*,*}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Rightarrow [X, Y]_*^G.$$

It is a finite spectral sequence concentrated in rows 0 to r (the rank of G) and strongly convergent for all X and Y . \square

This was what led us to attempt to prove the main theorem of the present paper, and many of the methods used to construct the Adams spectral sequence are adapted to the present work. Nonetheless, it appears that the only way we explicitly use the Adams spectral sequence is in the fact that cells are characterized by their homology.

Corollary 2.5. [18, 12.1] *If X is a G -spectrum with $\pi_*^{\mathcal{A}}(X) \cong \pi_*^{\mathcal{A}}(G/H_+)$ then $X \simeq G/H_+$.*

The proof proceeds by giving an explicit resolution of $\pi_*^{\mathcal{A}}(G/H_+)$ in $\mathcal{A}(G)$, and then observing that this gives appropriate vanishing at the E_2 -page so as to ensure an isomorphism $\pi_*^{\mathcal{A}}(X) \cong \pi_*^{\mathcal{A}}(G/H_+)$ lifts to a homotopy class of maps $G/H_+ \rightarrow X$. Since $\pi_*^{\mathcal{A}}$ detects weak equivalences, this suffices. Evidently, this argument applies in any model category with a similar Adams spectral sequence.

In the present paper, we often need to know how our chosen cells behave under functors between model categories. We will apply the corollary repeatedly to see that each cell maps to the obvious object up to equivalence.

3. COCHAIN RING SPECTRA

The purpose of this section is to discuss various choices of models for spectra.

3.A. Spectra. We will work in the category of orthogonal G -spectra [36]. Our present proof is not especially sensitive to which model we use, but the transition from spectra to DGAs is a little simpler if we use orthogonal spectra.

3.B. The sphere spectrum. Just as abelian groups are \mathbb{Z} -modules, giving \mathbb{Z} a special role, so too spectra are modules over the sphere spectrum \mathbb{S} . Although \mathbb{S} is the suspension spectrum of S^0 , we will generally use the special notation \mathbb{S} to emphasize its special role.

We have already declared that we are working rationally, so that \mathbb{S} will denote the rational sphere spectrum. There are various constructions of \mathbb{S} , but we choose a localization in the category of commutative ring spectra.

3.C. Choice of coefficients. Central to our formalism is that we consider ‘rings of functions’ on certain spaces, and then consider modules over these. In effect we take a suitable model for cochains on the space with coefficients in a ring. The purpose of the present subsection is to describe the options, and explain why we end up simply using the functional dual $DX = F(X, \mathbb{S})$ rather than one of the natural alternatives.

If X is a G -space and k is a ring G -spectrum then we may write

$$C^*(X; k) := D_k X := F_{\mathbb{S}}(X, k)$$

for the G -spectrum of functions from X to k . The first notation comes from the special case of an Eilenberg-MacLane spectrum, which gives a model for cohomology. The second notation comes from the special case $k = \mathbb{S}$ of the functional dual. This spectrum has a ring structure using the multiplication on k and the diagonal map of X . If k is a commutative ring spectrum then so is $C^*(X; k)$.

There are a number of related ring spectra of this form, and we briefly discuss their properties before explaining which is most relevant to us.

First, there is the rational sphere G -spectrum \mathbb{S} , and then there are two Eilenberg-MacLane G -spectra associated to Green functors. The first Green functor is the Burnside functor \mathbb{A} , whose value on G/H is the Burnside ring of H , and the second Green functor is the constant functor \mathbb{Q} .

To start with we observe that there are maps

$$\mathbb{S} \longrightarrow H\mathbb{A} \longrightarrow H\mathbb{Q}$$

of commutative ring G -spectra where the first map kills higher homotopy groups and the second kills the augmentation ideal. This induces maps

$$D_{\mathbb{S}}X \longrightarrow D_{H\mathbb{A}}X \longrightarrow D_{H\mathbb{Q}}X.$$

These are very far from being equivalences in general. For the second map that is clear since $\mathbb{A}(G/H) \neq \mathbb{Q}$ if H is a non-trivial finite subgroup. For the first, it is clear from the fact that \mathbb{S} has non-trivial higher homotopy (even rationally) when G is not finite.

Lemma 3.1. *(i) If X is free, the above maps induce equivalences*

$$D_{\mathbb{S}}X \simeq D_{H\mathbb{A}}X \simeq D_{H\mathbb{Q}}X.$$

(ii) If X has only finite isotropy, then the first map is an equivalence

$$D_{\mathbb{S}}X \simeq D_{H\mathbb{A}}X.$$

Proof: For Part (i) we note that \mathbb{S} , $H\mathbb{A}$ and $H\mathbb{Q}$ all have non-equivariant homotopy \mathbb{Q} in degree 0.

For Part (ii), \mathbb{S} is (rationally) an Eilenberg-MacLane spectrum for any finite group of equivariance. \square

The functor $D_{H\mathbb{Q}}$ has the convenient property that

$$(D_{H\mathbb{Q}}X)^G = D_{H\mathbb{Q}}(X/G)$$

for any space X . On the other hand, this lets us calculate values which show the functor is not the one we want to use (specifically, the homotopical analysis of [18] makes clear that the homotopy groups of the cochains on $E\mathcal{F}_+$ should be those of $D_{\mathbb{S}}E\mathcal{F}_+$). Since we will in fact only apply the duality functor to spaces with finite isotropy, we could equally well apply the functor $D_{H\mathbb{A}}$.

3.D. Choice of notation. Perhaps the most natural choice would be the notation $C^*(X; \mathbb{A})$. One argument against this is that some readers may be misled into thinking of a DGA, but in fact we avoided the notation on the grounds of brevity. We have chosen to write $DX = D_{\mathbb{S}}X = F(X, \mathbb{S})$ since the notation for the functional dual is well-known and suggestive.

3.E. Localizations. We now have commutative ring spectra $DE\mathcal{F}_+$, and we also wish to consider various spectra $DE\mathcal{F}_+ \wedge S^{\infty V(H)}$ as commutative ring spectra, where

$$S^{\infty V(H)} = \bigcup_{V^H=0} S^V.$$

Some care is necessary at this point.

It is clear that $S^{\infty V(H)}$ is a commutative ring up to homotopy, but it is not a strictly commutative ring for reasons described by [35] (or because it is incompatible with the existence of multiplicative norm maps). However it is not hard to provide a substitute for this structure. A G -spectrum is an $S^{\infty V(H)}$ -module up to homotopy if and only if it lies over H in the sense that it is K -contractible if $K \not\supseteq H$ (Subsection 11.D gives a more extended discussion). Similarly, any map of G -spectra over H is compatible up to homotopy with the action. Thus the category of G -spectra over H provides a well behaved substitute for the category of $S^{\infty V(H)}$ -modules, and we will use the language of modules without comment. From the point of view of model structures, the category of spectra over H is obtained from all G -spectra by localization to invert the maps which become equivalences on smashing with $S^{\infty V(H)}$.

On the other hand, we actually need to know that the ring structure on $DE\mathcal{F}_+ \wedge S^{\infty V(H)}$ arising as the smash product has a strictly commutative model. To give a commutative model we start by noting that $DE\mathcal{F}_+ \wedge S^{\infty V(H)}$ is the Bousfield localization of $DE\mathcal{F}_+$ with respect to the $DE\mathcal{F}_+$ -module $DE\mathcal{F}_+ \wedge S^{\infty V(H)}$. Next, we apply [11, VIII.2.2] which states that we may Bousfield localize a commutative ring spectrum R with respect to an R -cell module M to obtain a commutative ring spectrum R_M . Using [36, IV.1.5] we may move between orthogonal spectra and EKMM spectra. Non-equivariantly, the condition that M should be an R -cell module is no essential restriction, but equivariantly it is significant, since modules $R \wedge G/H_+$ need not be built from R . In particular this explains why the result cannot be applied to $R = \mathbb{S}$ and $M = S^{\infty V(H)}$.

It therefore remains to explain that $DE\mathcal{F}_+ \wedge S^{\infty V(H)}$ is a $DE\mathcal{F}_+$ -cell module. Indeed, it is a homotopy direct limit of spectra $DE\mathcal{F}_+ \wedge S^V$, so the result follows from a lemma.

Lemma 3.2. *For any complex representation V , the spectrum $DEF_+ \wedge S^V$ is a finite wedge of suspensions of retracts of DEF_+ . In particular, $DEF_+ \wedge S^V$ is equivalent to a DEF_+ -cell module.*

Proof: The proof is based on the Thom isomorphism, which in turn comes from the splitting $E\mathcal{F}_+ \simeq \bigvee_F E\langle F \rangle$ [12] which immediately gives a splitting of the dual as a product.

Now suppose given a representation V and use the Thom isomorphism

$$S^{-V} \wedge E\langle F \rangle \simeq S^{-|V^F|} \wedge E\langle F \rangle$$

for each finite subgroup F [18]. We divide the finite subgroups into sets according to $\dim_{\mathbb{C}}(V^F)$. Of course there are only finitely many of these, and we may apply the corresponding orthogonal idempotents to consider the sets separately.

For the class of F with $\dim_{\mathbb{C}}(V^F) = k$ we have

$$\prod_F D(E\langle F \rangle \wedge S^{-V}) \simeq \Sigma^{-k} \prod_F D(E\langle F \rangle),$$

which is a suspension of a retract of DEF_+ . □

Finally, we will need to compare these two approaches to providing models, and we observe that they agree in the context where they both apply. In fact localization gives a ring homomorphism $l : DEF_+ \rightarrow DEF_+ \wedge S^{\infty V(H)}$, and extension and restriction of scalars give a Quillen pair

$$l_* : DEF_+ \text{-mod-} G\text{-spectra} \rightleftarrows DEF_+ \wedge S^{\infty V(H)} \text{-mod-} G\text{-spectra} : l^* .$$

By the universal property of localization, this gives a new Quillen pair which is in fact a Quillen equivalence

$$DEF_+ \text{-mod-} G\text{-spectra} / H \simeq DEF_+ \wedge S^{\infty V(H)} \text{-mod-} G\text{-spectra}$$

where the model category on the left is the localization of $DEF_+ \text{-mod-} G\text{-spectra}$ in which maps which become an equivalence on smashing with $S^{\infty V(H)}$ have been inverted. This construction becomes important in Section 9.

4. HASSE MODELS

Our overall strategy is to assemble a model for all G -spectra from models for G -spectra with geometric isotropy K as K ranges over all closed subgroups. In fact we will collect together information from all the isotropy groups with the same identity component, so the pieces to be assembled are indexed by the connected subgroups of G . We will describe the particular ingredients in more detail later in the spectral setting, but in this section we describe the general process of assembly in an algebraic setting. Since this algebraic setting is not necessary for the rest of the paper, we will not give full details.

4.A. The classical case. The Hasse square lets us recover \mathbb{Z} from the pullback square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & (\prod_p \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \end{array}$$

Applying the same process allows us to recover any abelian group from the pullback square

$$\begin{array}{ccc} M & \xrightarrow{\quad} & M \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ M \otimes \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & M \otimes (\prod_p \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \end{array}$$

If M is finitely generated this can be obtained by completing and rationalizing M , but to get a pullback square for all M , the construction must be obtained by tensoring the square for the ring with M .

This may be viewed as a means of reconstructing the entire category of abelian groups from the category of diagrams of modules over the diagram of rings

$$\begin{array}{ccc} & \mathbb{Q} & \\ & \downarrow & \\ \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & (\prod_p \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}. \end{array}$$

4.B. General context. We want to apply this method more generally so we proceed as follows. Suppose given a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R^l \\ \beta \downarrow & & \downarrow \gamma \\ R^c & \xrightarrow{\delta} & R^t. \end{array}$$

Delete R and consider the diagram

$$R^\perp = \left(\begin{array}{ccc} & & R^l \\ & & \downarrow g \\ R^c & \xrightarrow{d} & R^t \end{array} \right)$$

with three objects. We may form the category $R^\perp\text{-mod}$ of diagrams

$$\begin{array}{ccc} & M^l & \\ & \downarrow g & \\ M^c & \xrightarrow{d} & M^t \end{array}$$

where M^l is an R^l -module, M^c is an R^c -module, M^t is an R^t -module and the maps g and d are module maps over the corresponding maps of rings. (See also Section 5.A.) Since R^\perp is a diagram of R -algebras, termwise tensor product gives a functor

$$R^\perp \otimes_R : R\text{-mod} \longrightarrow R^\perp\text{-mod}.$$

Similarly, since R maps to the pullback PR^\perp , pullback gives a functor

$$P : R^\perp\text{-mod} \longrightarrow R\text{-mod}.$$

It is easily verified that these give an adjoint pair

$$R^\perp \otimes_R : R\text{-mod} \rightleftarrows R^\perp\text{-mod} : P.$$

We may then consider the unit

$$\eta : M \longrightarrow P(R^\perp \otimes_R M),$$

and the first condition for it to be a natural isomorphism is that it should be so when $M = R$, which is to say the original diagram of rings is a pullback. It is quite easy to identify sufficient conditions for η to be an isomorphism in general. First we require that the diagram is a pushout of modules, so that there is a long exact Tor sequence, and second that R^t is a flat R -module so that the sequence is actually short.

On the other hand, we cannot expect the counit of the adjunction to be an equivalence since we can add any module to M^t without changing PM^\perp . Accordingly, we need to find a way to focus attention on diagrams arising from actual R -modules.

4.C. Model structures. We now suppose that the commutative diagram given above is a diagram of DGAs and use $R\text{-mod}$ to denote the category of DG R -modules. We give it the (algebraically) projective model structure, with homology isomorphisms as weak equivalences and fibrations the surjections. The cofibrations are retracts of relative cell complexes, where the spheres are shifted copies of R . The diagram category $R^\perp\text{-mod}$ gets the diagram-injective model structure in which cofibrations and weak equivalences are maps which have this property objectwise; the fibrant objects have γ and δ surjective. This diagram-injective model structure is shown to exist for ring spectra in Theorem B.1, and the same proof works for DGAs.

Since extension of scalars is a left Quillen functor for the (algebraically) projective model structure for any map of DGAs, $R^\perp \otimes_R -$ also preserves cofibrations and weak equivalences and is therefore also a left Quillen functor. We then apply the Cellularization Principle (Appendix A) to obtain the following result.

Proposition 4.1. *Assume given a commutative square of DGAs which is a homotopy pullback. The adjunction induces a Quillen equivalence*

$$R\text{-mod} \xrightarrow{\cong} \text{cell-}R^\perp\text{-mod},$$

where cellularization is with respect to the image, R^\perp , of the generating R -module R .

Proof: We apply Proposition A.6, which states that if we cellularize the model categories with respect to corresponding sets of objects, we obtain a Quillen equivalence. The argument is described in greater generality and more detail in Proposition B.4.

In the present case, we cellularize with respect to the single R -module R on the left, and the corresponding diagram R^\perp on the right. Since the original diagram of rings is a homotopy pullback, the unit of the adjunction is an equivalence for R , and we see that the generator R and the generator R^\perp correspond under the equivalence, as required in the hypothesis in Part (2) of Proposition A.6.

Since R is cofibrant and generates $R\text{-mod}$, cellularization with respect to R has no effect on $R\text{-mod}$ and we obtain the stated equivalence with the cellularization of $R^\perp\text{-mod}$ with respect to the diagram coming from R . \square

4.D. The case of the circle. It is worth introducing our results on G -spectra with the special case of the circle group $G = T$ because the diagram of ring spectra is just as simple as the classical case of the Hasse square considered in Subsection 4.A. Indeed, the diagram analogous to the starting diagram of rings is

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \tilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}, \end{array}$$

although $\tilde{E}\mathcal{F}$ is not in fact a strictly commutative ring. This is a homotopy pullback square because the map between the fibres of the two horizontals is $S^0 \wedge E\mathcal{F}_+ \longrightarrow DE\mathcal{F}_+ \wedge E\mathcal{F}_+$, which is an equivalence since $E\mathcal{F}_+ \longrightarrow S^0$ is an \mathcal{F} -equivalence. We may replace this diagram by a diagram in which the horizontal is a fibration of ring T -spectra, and the vertical is a fibration, and we write \tilde{R}_{top}^j for the resulting diagram

$$\begin{array}{ccc} & & \tilde{E}\mathcal{F} \\ & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}. \end{array}$$

Making explicit the adaption for the fact that $\tilde{E}\mathcal{F}$ is not a ring, an \tilde{R}_{top}^j -module is a diagram

$$\begin{array}{ccc} & & M^t \\ & & \downarrow \\ M^c & \longrightarrow & M^t. \end{array}$$

where M^c is a $DE\mathcal{F}_+$ -module, M^t is a $DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}$ -module, $M^c \longrightarrow M^t$ is a map over $DE\mathcal{F}_+ \longrightarrow DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}$, and in fibrant diagrams M^l is \mathcal{F} -contractible. The existence of a diagram-injective type model structure on this category of modules is established in Theorem B.1. The discussion proceeds exactly as in the algebraic case, but when we cellularize we must use a generating set, so we use the cells T/H_+ as H runs through all closed subgroups of T .

Proposition 4.2. *The adjunction induces a Quillen equivalence*

$$T\text{-spectra} \xrightarrow{\simeq} \text{cell-}\tilde{R}_{top}^\bullet\text{-mod-}G\text{-spectra}.$$

Proof: The proof precisely follows the algebraic case (Proposition 4.1). To see that the unit is an equivalence for all cells G/H_+ (and not just for $S^0 = G/G_+$), we observe that smashing with G/H_+ preserves homotopy pullback squares. \square

Part 2. Formalities with diagrams

5. DIAGRAMS OF RINGS AND MODULES

Throughout this paper we consider categories of modules over diagrams of rings in two contexts: differential graded modules over DGAs and module spectra over ring spectra. In

this section we introduce the relevant model structures and begin the discussion of cells and changing diagram shapes. In the last subsection, 5.F, we briefly outline how these concepts figure in the the following sections.

5.A. The archetype. Given a diagram shape \mathbf{D} , we may consider a diagram $R : \mathbf{D} \rightarrow \mathbb{C}$ of rings in a category \mathbb{C} . We may then consider the category of R -modules. These will be diagrams $M : \mathbf{D} \rightarrow \mathbb{C}$ in which $M(x)$ is an $R(x)$ -module for each object x , and for every morphism $a : x \rightarrow y$ in \mathbf{D} , the map $M(a) : M(x) \rightarrow M(y)$ is a module map over the ring map $R(a) : R(x) \rightarrow R(y)$.

Each map $R(a) : R(x) \rightarrow R(y)$ gives rise to a restriction functor

$$a^* : R(y)\text{-mod} \rightarrow R(x)\text{-mod},$$

which in turn has both a left adjoint

$$R(y)\text{-mod} \xleftarrow{a_*} R(x)\text{-mod},$$

defined by $a_*(M_x) = R(y) \otimes_{R(x)} M_x$ and a right adjoint

$$R(y)\text{-mod} \xleftarrow{a_!} R(x)\text{-mod},$$

defined by $a_!(M_x) = \text{Hom}_{R(x)}(R(y), M_x)$.

Accordingly, when considering an R -module, there are two ways to view the structure map $M(a) : M(x) \rightarrow M(y)$. First we can view it as a map $M(a) : M(x) \rightarrow a^*M(y)$ of $R(x)$ -modules (the *restriction* model). Second we can consider its left adjoint

$$\widetilde{M}(a) : R(y) \otimes_{R(x)} M(x) = a_*M(x) \rightarrow M(y)$$

as a map of modules over the ring $R(y)$ (the *extension of scalars* model).

5.B. A generalization. When considering diagrams of equivariant spectra, we will need to use a slight generalization of the archetype. Indeed, we begin with a diagram $\mathbb{M} : \mathbf{D}^{op} \rightarrow \text{Cat}$ of categories (the previous special case is $\mathbb{M}(x) = R(x)\text{-mod}$). Associated to each map $a : x \rightarrow y$ we have a functor $a^* : \mathbb{M}(y) \rightarrow \mathbb{M}(x)$. We may then consider the category of \mathbb{M} -diagrams, whose objects consist of an object $M(x)$ from $\mathbb{M}(x)$ for each object x of \mathbf{D} and a transitive system of morphisms

$$M(a) : M(x) \rightarrow a^*M(y)$$

for each morphism $a : x \rightarrow y$ in \mathbf{D} (the *right adjoint* form). If a^* has a left adjoint a_* , we may consider the adjunct

$$\widetilde{M}(a) : a_*M(x) \rightarrow M(y)$$

(the *left adjoint* form).

5.C. Model structures. If we suppose that each category $\mathbb{M}(x)$ has a model structure, there are two ways to attempt to put a model structure on the category of \mathbb{M} -diagrams $\{M(x)\}_{x \in \mathbf{D}}$. The diagram-projective model structure (if it exists) has its fibrations and weak equivalences defined objectwise. The diagram-injective model structure (if it exists) has its cofibrations and weak equivalences defined objectwise. It must be checked in each particular case whether or not these specifications determine a model structure. When both model structures exist, it is clear that the identity functors define a Quillen equivalence between them.

Provided we require that the adjoint pair $a_* \vdash a^*$ of functors relating the model categories form a Quillen pair, for all diagrams that we consider Theorem B.1 proves that both the diagram projective and diagram-injective model structures exist.

5.D. Simple change of diagram equivalences. Returning to a diagram of modules over a diagram of rings, it often happens that if we specify the modules on just part of the diagram we can fill in the remaining entries in a canonical way. We will distinguish two types of examples: (1) where the diagram is filled in by extension of scalars (a left adjoint) and (2) where the diagram is filled in using a pullback (a right adjoint). In either case, this sometimes induces an equivalence between subcategories of modules over the larger and smaller diagrams.

Example 5.1. The simplest example of (1) is the diagram $R = (R_1 \longrightarrow R_2)$ of rings. An R -module gives rise to an R_1 -module by evaluation at the first object. An R_1 -module M_1 give rise to the R -module

$$(R \otimes_{R_1} M_1) = (R_1 \otimes_{R_1} M_1 \longrightarrow R_2 \otimes_{R_1} M_1).$$

The discussion of Hasse squares in Section 4 gives a good example of (2).

More precisely, we suppose $i : \mathbf{D} \longrightarrow \mathbf{E}$ is the inclusion of a subdiagram, and that $R : \mathbf{E} \longrightarrow \mathbb{C}$ is a diagram of rings. We may restrict R to a diagram $R|_{\mathbf{D}} : \mathbf{D} \longrightarrow \mathbb{C}$, and this gives rise to a restriction functor

$$i^* : R\text{-mod} \longrightarrow R|_{\mathbf{D}}\text{-mod}.$$

We discuss two cases, depending whether i^* is a right or left adjoint.

Case R. If i^* is a right adjoint with left adjoint i_* , and if there are diagram-projective model structures (with objectwise weak equivalences and fibrations) on the two categories, the adjunction (i_*, i^*) is a Quillen pair. Note that i_* necessarily leaves the entries in \mathbf{D} unchanged, which is to say that the derived unit $M \longrightarrow i^*i_*M$ is an equivalence.

Next, we suppose that we have a set \mathcal{D} of cells in the category of $R|_{\mathbf{D}}$ -modules. Given the existence of a left adjoint, this gives rise to a set $i_*\mathcal{D}$ of cells in the category of R -modules. This automatically means that the collection of cells \mathcal{D} coincides with the restrictions of the induced cells $i^*i_*\mathcal{D}$; in practice we often specify a set \mathcal{E} of cells σ in the category of R -modules and require that the derived counit $i_*i^*\sigma \longrightarrow \sigma$ is a weak equivalence for all σ , where the two constructions are related by $\mathcal{D} = i^*\mathcal{E}$ and $\mathcal{E} = i_*\mathcal{D}$; we say that the set of cells \mathcal{E} is *induced from* \mathcal{D} . If we use cells \mathcal{D} for \mathbf{D} and \mathcal{E} for \mathbf{E} with \mathcal{E} induced from \mathcal{D} , the Cellularization Principle A.6 shows the original Quillen pair determines a Quillen equivalence on cellularized categories. Note, we are using Remark A.3 here to see that the cellularized model category depends only on the weak equivalence type of the chosen cells. Usually, we have particular sets of cells implicit, and if those in \mathbf{E} are induced from those of \mathbf{D} we say that \mathbf{D} is *right cellularly dense* in \mathbf{E} .

Note that to check that \mathbf{D} is right cellularly dense in \mathbf{E} we need to check the derived counit $i_*i^*\sigma \longrightarrow \sigma$ is a weak equivalence for each cell $\sigma \in \mathcal{E}$. Because we are using the diagram projective model structure, this is checked by verifying a condition at each point in the diagram, and this is automatic at points of \mathbf{D} , so we only need to verify that i_* fills in the values of σ correctly at points of \mathbf{E} that are not in \mathbf{D} .

Case L. Similarly if i^* is a left adjoint with right adjoint, $i_!$, and both categories have diagram-injective model structures (with objectwise weak equivalences and cofibrations) then the adjunction $(i^*, i_!)$ is a Quillen pair. Note that $i_!$ necessarily leaves the entries in \mathbf{D} unchanged, which is to say that the derived counit $i^*i_!M \rightarrow M$ is an equivalence.

This time we use the right adjoint to specify a set $i_!\mathcal{D}$ of cells in the category of R -modules. This automatically means that the collection of cells \mathcal{D} coincides with the restrictions of the induced cells $i^*i_!\mathcal{D}$; in practice we often specify a set \mathcal{E} of cells σ in the category of R -modules and require that the derived unit $\sigma \rightarrow i_!i^*\sigma$ is a weak equivalence for all σ , where the two constructions are related by $\mathcal{D} = i^*\mathcal{E}$ and $\mathcal{E} = i_!\mathcal{D}$; we say that the set of cells \mathcal{E} is *extended from \mathcal{D}* . If we use cells \mathcal{D} for \mathbf{D} and \mathcal{E} for \mathbf{E} with \mathcal{E} extended from \mathcal{D} , the Cellularization Principle A.6 shows the original Quillen pair determines a Quillen equivalence on cellularized categories; see also Remark A.3. Usually, we have particular sets of cells implicit, and if those in \mathbf{E} are extended from those of \mathbf{D} we say that \mathbf{D} is *left cellularly dense* in \mathbf{E} .

Note that to check that \mathbf{D} is left cellularly dense in \mathbf{E} we need to check the derived counit $\sigma \rightarrow i_!i^*\sigma$ is a weak equivalence for each cell $\sigma \in \mathcal{E}$. Because we are using the diagram injective model structure, this is checked by verifying a condition at each point in the diagram, and this is automatic at points of \mathbf{D} , so we only need to verify that $i_!$ fills in the values of σ correctly at points of \mathbf{E} that are not in \mathbf{D} .

5.E. Change of diagram equivalences. If there is a chain

$$\mathbf{D} = \mathbf{D}_0 \subseteq \mathbf{D}_1 \subseteq \cdots \subseteq \mathbf{D}_n = \mathbf{E}$$

of diagrams, with specified sets of cells in their diagram categories, and if at each step i^* is either left or right cellularly dense, we say that $\mathbf{D} \rightarrow \mathbf{E}$ is a **dense inclusion**.

In this case most of the terms will be involved in both a left and a right dense inclusion, and therefore will be required to have both a projective and an injective model structure. These two model structures are equivalent by using the identity functor. Using these, and the Quillen equivalences from the simple cellularly dense inclusions, we obtain a Quillen equivalence

$$\text{cell-}R|_{\mathbf{D}\text{-mod}} \simeq \text{cell-}R|_{\mathbf{E}\text{-mod}}.$$

Some of the restrictions are left Quillen functors and some are right Quillen functors, so this new equivalence is given by a zig-zag of simple Quillen equivalences.

Our overall strategy for giving Quillen equivalences of diagram categories of modules will be to find a diagram \mathbf{E} and two subcategories $\mathbf{D}_1, \mathbf{D}_2$, so that the maps

$$\mathbf{D}_1 \rightarrow \mathbf{E} \leftarrow \mathbf{D}_2$$

are both dense inclusions. This in turn will establish Quillen equivalences

$$\text{cell-}R|_{\mathbf{D}_1\text{-mod}} \simeq \text{cell-}R|_{\mathbf{E}\text{-mod}} \simeq \text{cell-}R|_{\mathbf{D}_2\text{-mod}}.$$

5.F. The strategy. We briefly explain how these ideas will be applied. There are a number of details that need to be explained, so this brief subsection necessarily omits many essential details.

We begin in topology with the category of G -spectra as modules over the (single object) ring G -spectrum \mathbb{S} . By extension of scalars we extend over a diagram $\mathbf{D}_1 = \mathbf{LC}(G)$ in which the single object is initial. This is introduced in Subsection 6.B; \mathbf{L} stands for localization and \mathbf{C} for completion. Next \mathbf{D}_1 is cellularly dense in a large diagram $\mathbf{E} = \mathbf{LCI}(G)$ introduced

in Subsection 6.D, the letters **L** and **C** stand for localization and completion as before, and **I** stands for inflation. There is a second cellularly dense diagram $\mathbf{D}_2 = \mathbf{LI}(G)$ discussed in Subsection 6.E, with **L** and **I** standing for localization and inflation as before. This shows that the category of rational G -spectra is equivalent to cellular objects in a category $\text{cell-}\tilde{R}_{\text{top}}^{\mathbf{LI}}\text{-mod-}G\text{-spectra}$ of equivariant module spectra. This diagram of ring G -spectra is then of a suitable form to be compared, first (Theorem 13.2) to a diagram of non-equivariant ring spectra and then (in Section 14) to a diagram of DGAs.

The details will be filled in thoroughly: in Section 6 we introduce the diagrams we work with, in Section 7 it is explained that the category $\mathcal{A}(G)$ can be viewed as such a category of modules. After introducing some relevant equivariant homotopy theory and a diagram of G -spectra, we return in Section 10 to give details of the density arguments.

6. DIAGRAMS OF QUOTIENT GROUPS

As is usual in equivariant topology, we need to organize information associated to fixed point sets under closed subgroups, and this is done using various diagrams of subgroups. Although we could get away with simpler diagrams to begin with, it is much easier to explain the arguments if we have a consistent framework from the beginning. Accordingly, we spend this section introducing the diagrams $\mathbf{LC}(G)$, $\mathbf{LI}(G)$ and $\mathbf{LCI}(G)$ mentioned in Subsection 5.F. These will be used to index our rings with many objects and associated categories of modules.

We motivate the categories from the starting point of equivariant topology which will explain the words ‘localization’, ‘completion’ and ‘inflation’ and hence the names of the diagrams. From the point of view of equivariant topology, $\mathbf{LC}(G)$ seems sufficient in itself, but it is harmless to extend the diagram to $\mathbf{LCI}(G)$. Once this is done, we may describe the second subdiagram $\mathbf{LI}(G)$ needed in algebra (the reinterpretation of $\mathcal{A}(G)$ in these terms is described in Section 7).

6.A. Connected subgroups. Recall that our basic objects of study are G -equivariant cohomology theories, where G is a torus. The usual philosophy is that one should break down problems by isotropy groups.

The first simplification is that because we are working rationally, it is possible to treat all subgroups with a fixed identity component K at the same time. Typically the subgroups with identity component K do not interact with each other, and typically they all behave much the same.

The first consequence of this is to focus attention on the partially ordered set $\mathbf{ConnSub}(G)$ of *connected* subgroups of G . The second is that it is convenient to have the notation \mathcal{F} for the set of finite subgroups, and more generally, \mathcal{F}/K for the set of subgroups with identity component K .

6.B. The calculation scheme. When building a model of G -spectra we may imagine that we have already built a model for K -fixed point objects whenever K is a non-trivial connected subgroup (as above, this also provides the information for fixed points under all subgroups with identity component K). The point here is that G/K is then a torus of lower dimension, which can be assumed understood by induction. This leads us to consider the poset $\mathbf{ConnSub}(G)$, where the non-trivial subgroups correspond to existing information, and the trivial subgroup to new information. Because the subgroup K indexes G/K -equivariant

information it is clearer to rename the objects and consider the poset $\mathbf{ConnQuot}(\mathbf{G})$ of quotients of G by connected subgroups, where the maps are the quotient maps.

Because this is fundamental, we standardize the display so that $\mathbf{ConnQuot}(\mathbf{G})$ is arranged horizontally (i.e., in the x -direction), with quotients decreasing in size from $G/1$ at the left to G/G at the right. When G is the circle $\mathbf{ConnQuot}(\mathbf{G}) = \{G/1 \rightarrow G/G\}$ but when G is a torus of rank ≥ 2 it has a unique initial object 1, a unique terminal object G , but infinitely many objects at every other level. This makes it harder to draw $\mathbf{ConnQuot}(\mathbf{G})$, so we will usually draw a diagram by choosing one or two representatives from each level, so that in rank r , the diagram $\mathbf{ConnSub}(\mathbf{G})$ is illustrated by

$$G/1 \rightarrow G/H_1 \rightarrow G/H_2 \rightarrow \cdots \rightarrow G/H_{r-1} \rightarrow G/G$$

where H_i is an i -dimensional subtorus of G .

This leaves the finite subgroups, which provide new information, which must be understood by other means. In fact this will amount to a generalization of the traditional method used for free spaces, namely the Borel construction. We represent this by the diagram $\mathbf{2} = \{n \rightarrow c\}$, where we have chosen the letters n for ‘natural’ and c for ‘complete’; we will use the letter a to indicate an unspecified choice of n and c . Here we will arrange the diagram in the y -direction with n above c .

This suggests we should consider the diagram

$$\mathbf{LC}(G) := \mathbf{ConnQuot}(\mathbf{G}) \times \{n \rightarrow c\}.$$

This is the basic array on which we represent all our information, so it is worth displaying the rank 2 case. In this picture x is horizontal and, as is usual when we only have x and y axes to draw, y is up the page

$$\begin{array}{ccccc} & (G/1, n) & \longrightarrow & (G/H, n) & \longrightarrow & (G/G, n) \\ & \downarrow & & \downarrow & & \downarrow \\ \begin{array}{c} \uparrow y \\ \downarrow x \end{array} & (G/1, c) & \longrightarrow & (G/H, c) & \longrightarrow & (G/G, c) \end{array}$$

Later, we will have x , y and z axes. Again we will use the usual convention: x will remain horizontal, y will be into the page, and z will be vertical.

In effect the topological calculation scheme will start with the input data consisting of the c -row and the natural G -fixed entry $(G/G, n)$, and reconstruct the rest of the entire from that. This is similar to what was done in Subsection 4.D for the circle group T .

6.C. Inflation. Although the information at the vertices has come from quotients of G , the diagram $\mathbf{LC}(G)$ will typically be in a G -equivariant category. It is therefore convenient to extend the diagram in the z -direction by adding copies of the diagrams $\mathbf{LC}(G/L)$ for connected subgroups L . The $z = 0$ level corresponds to $G = G/1$ itself, and for each connected subgroup L , the G/L -equivariant information is represented at a suitable z level. It is helpful to think of placing the G/L information at $z = n$ when L is of dimension n .

Of course the information above the point $(G/H, a)$ will only be relevant at z -levels G/K where $K \subseteq H$; we write $(G/H, a)_{G/K}$ for this point. The arrows are inflations

$$v_{G/K}^{G/L} : (G/H, a)_{G/K} \rightarrow (G/H, a)_{G/L} \text{ when } L \subseteq K \subseteq H \subseteq G;$$

these are thought of as *vertical* since they change the z -coordinate. Note that in each such vertical column, there is a unique lowest entry $(G/H, a)_{G/1}$, and a unique highest entry $(G/H, a)_{G/H}$.

6.D. The full diagram. The entire diagram $\mathbf{LCI}(G)$ has objects $(G/K, a)_{G/L}$ where $a \in \{n, c\}$ and $L \subseteq K$ is an inclusion of connected subgroups of G . There are three types of morphisms. The horizontal maps (increasing the x coordinate)

$$h_K^H : (G/K, a)_{G/L} \longrightarrow (G/H, a)_{G/L} \text{ where } L \subseteq K \subseteq H \subseteq G,$$

vertical maps (decreasing the z coordinate)

$$v_L^K : (G/H, a)_{G/K} \longrightarrow (G/H, a)_{G/L} \text{ where } L \subseteq K \subseteq H \subseteq G$$

and completions (changing the y coordinate)

$$\kappa : (G/K, n)_{G/L} \longrightarrow (G/K, c)_{G/L}.$$

We will introduce an $\mathbf{LCI}(G)$ -diagram of rings in Section 9. The fact that the horizontal maps are localizations, the vertical maps are inflations and the maps κ are completions explains the names of the diagrams.

6.E. The two subdiagrams. By construction we have already displayed the subdiagram $\mathbf{LC}(G)$ as given by the objects $(G/H, a)_{G/1}$ in $\mathbf{LCI}(G)$ with subscript $G/1$; the diagram contains the h and κ structure maps corresponding to localization and completion. The resulting diagrams typically consist of G -equivariant objects, and we sometimes omit the subscript $G/1$ when the danger of confusion is small.

The subdiagram $\mathbf{LI}(G)$ consists of objects $(G/K, c)_{G/L}$ in $\mathbf{LCI}(G)$ corresponding to ‘complete’ objects; the diagram contains the h and v structure maps corresponding to localization and inflation. We sometimes abbreviate this to $(G/K)_{G/L}$ when the context avoids confusion. Occasionally we even omit subscripts $G/1$, reflecting the fact that the points $G/K = (G/K, c)_{G/1}$ are the most important points in the diagram.

Next we explain why $\mathbf{LC}(G)$ and $\mathbf{LI}(G)$ are cellularly dense in $\mathbf{LCI}(G)$. Indeed, we will display a sequence of elementary dense inclusions.

6.F. Density of $\mathbf{LC}(G)$. The density of $\mathbf{LC}(G)$ in $\mathbf{LCI}(G)$ is actually rather routine.

There are two types of extension. For clarity we may start from the diagram $\mathbf{LC}_0^*(G)$ with the single object $(G/1, n)_{G/1}$. Next, we fill in the rest of the $z = 0$ face to obtain $\mathbf{LC}_0(G) = \mathbf{LC}(G)$ itself.

We then repeat this layer by layer, so that $\mathbf{LC}_s^*(G)$ consists of all subdiagrams $\mathbf{LC}(G/K)$ for $\dim(K) < s$ together with $(G/H, n)_{G/H}$ for $\dim(H) = s$, and $\mathbf{LC}_s(G)$ consists of all subdiagrams $\mathbf{LC}(G/K)$ for $\dim(K) \leq s$.

This gives the inclusions

$$\mathbf{LC}_0^*(G) \subseteq \mathbf{LC}_0(G) \subseteq \mathbf{LC}_1^*(G) \subseteq \mathbf{LC}_1(G) \subseteq \cdots \subseteq \mathbf{LC}_r^*(G) \subseteq \mathbf{LC}_r(G) = \mathbf{LCI}(G).$$

There are two types of inclusion

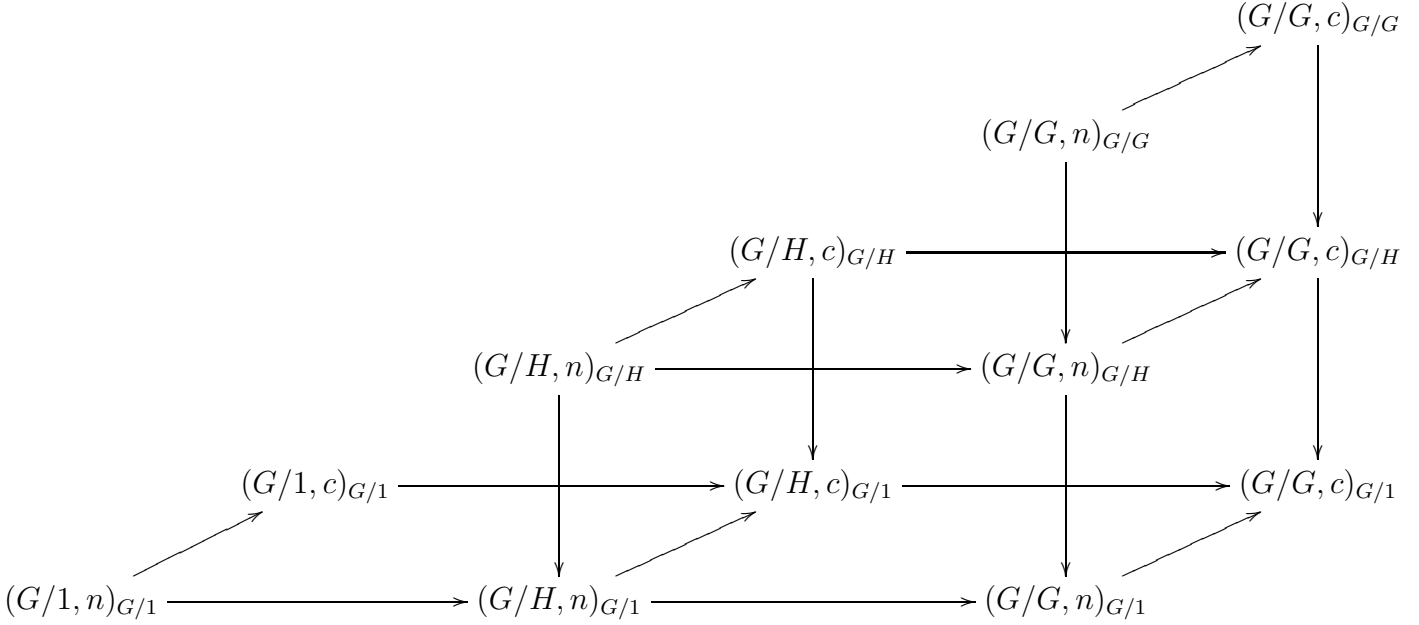
Type LC1: For each s , the extension from $\mathbf{LC}_s^*(G)$ to $\mathbf{LC}_s(G)$ is a straightforward extension of scalars. Thus restriction is a right adjoint, and we need to use diagram-projective model structures. This case is discussed in more detail in Subsection 10.C

Type LC2: For each s , the passage from $\mathbf{LC}_s(G)$ to $\mathbf{LC}_{s+1}^*(G)$ involves using additional structure (a fixed point functor) special to the equivariant situation. In this case restriction is

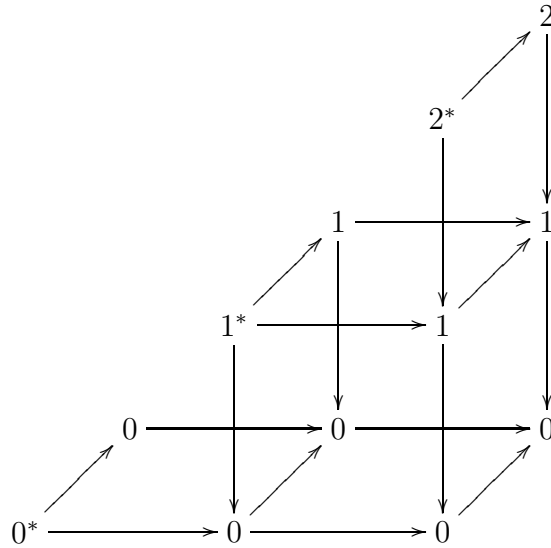
a left adjoint, and we need to use diagram-injective model structures. This case is discussed in more detail in Subsection 10.D

It may be helpful to illustrate the way diagrams are filled in with the rank 2 case.

Example 6.1. When G is of rank 2, the part of the diagram $\mathbf{LCI}(G)$ with a chosen circle H is as follows.



The order in which the entries are filled in is as illustrated below. Thus we start with the vertex marked 0^* . The vertices marked 0 on the plane $z = 0$ are filled in by extension of scalars. The vertex 1^* is filled in by using a fixed point functor from the 0 entry below it. The vertices marked 1 are filled in by extension of scalars in the $z = 1$ plane. The entry marked 2^* is filled in by using a fixed point functor from the entry 1 below it. Finally, the entry 2 is filled in by extension of scalars in the $z = 2$ plane.



6.G. Density of $\mathbf{LI}(G)$. The density of $\mathbf{LI}(G)$ in $\mathbf{LCI}(G)$ is a substantial and interesting result. In the general case, density is proved by the sequence of inclusions.

$$\mathbf{LI}^0 \subseteq \mathbf{LI}_*^0 \subseteq \mathbf{LI}^1 \subseteq \mathbf{LI}_*^1 \subseteq \cdots \subseteq \mathbf{LI}^r \subseteq \mathbf{LI}_*^r = \mathbf{LCI}(G).$$

Here $\mathbf{LI}^0(G) = \mathbf{LI}(G)$ consists of all the ‘complete’ entries with $y = c$, and $\mathbf{LI}_*^0(G)$ adds on $(G/G, n)_{G/G}$. At each subsequent stage $\mathbf{LI}^s(G)$ consists of all complete objects, together with all objects $(G/H, n)_{G/K}$ with $\text{codim}(K) < s$. The diagram $\mathbf{LI}_*^s(G)$ is obtained from $\mathbf{LI}^s(G)$ by adjoining all objects $(G/K, n)_{G/K}$ with $\text{codim}(K) = s$.

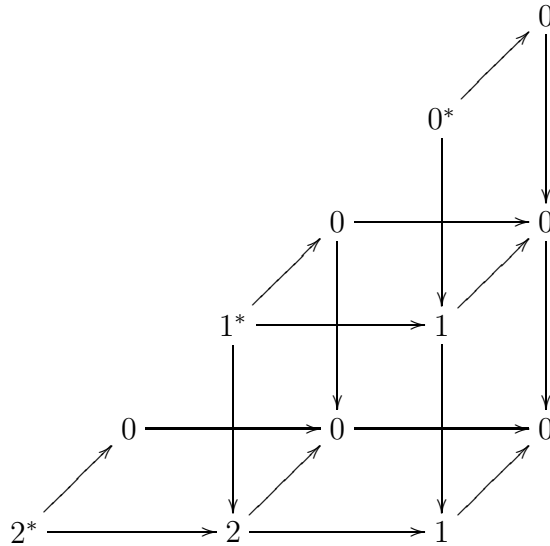
Type LI1: For each s , the extension from $\mathbf{LI}_*^{s-1}(G)$ to $\mathbf{LI}^s(G)$ is an extension of scalars. Thus restriction is a right adjoint, and we need to use diagram-projective model structures. This case is discussed in more detail in Subsection 10.E

Type LI2: For each s , the extension from $\mathbf{LI}^s(G)$ to $\mathbf{LI}_*^s(G)$ is given by a pullback or ‘continuous pullback’. Of the four cases, this is the one with most real content, and it requires considerable discussion.

In the simple pullback case, restriction is simply a left adjoint, and we need to use diagram-injective model structures. In the case of a continuous pullback, the inclusion needs to be further factorized as a composite of left and right Quillen functors. This case is discussed in more detail in Subsection 10.F

It may be helpful to illustrate the way diagrams are filled in with the rank 2 case.

Example 6.2. When G is of rank 2, we again work with the same part of the diagram $\mathbf{LCI}(G)$ with only one of the countable number of circle subgroups H displayed. The order in which the entries are filled in is as illustrated below. Thus we start with the vertices marked 0. The vertex marked 0^* is isomorphic to the 0 adjacent to it in the $z = 2$ plane. Now the vertex 1 below it is filled in by extension of scalars. The vertices 1^* are filled in by pullback on the $z = 1$ plane. Each entry marked 2 is filled in by extension of scalars along the vertical above it. Finally the entry 2^* is filled in by a certain ‘continuous pullback’ in the $z = 0$ plane.



7. THE CATEGORY $\mathcal{A}(G)$ AS A CATEGORY OF MODULES

For comparison with topological categories, we want to express the category $\mathcal{A}(G)$ as a category of modules over a diagram of rings. In the language of Section 6, we will introduce a diagram of rings based on the second subdiagram $\mathbf{LI}(G)$ of $\mathbf{LCI}(G)$ and show that $\mathcal{A}(G)$ is a category of modules over it.

7.A. The diagrams. We would like to relate the algebraic categories to the diagrams described in Section 6. Since the algebraic categories we consider consist of modules over the ring $\mathcal{O}_{\mathcal{F}}$ and its G/K -equivariant counterpart $\mathcal{O}_{\mathcal{F}/K}$ as defined in Section 2, we are really thinking of modules over completed rings. Accordingly, we only need the completed part of the diagram $\mathbf{LCI}(G)$, which is to say $\mathbf{LI}(G)$, the part consisting of objects $(G/H, c)_{G/K}$.

To permit readers to read this section immediately after Section 2 we will define this subdiagram $\mathbf{LI}(G)$ from scratch. We will abbreviate $(G/H, c)_{G/K}$ to $(G/H)_{G/K}$ and sometimes also abbreviate $(G/H)_{G/1}$ further to G/H . We follow the organization by x , y and z axis introduced in Subsection 6.D above.

Definition 7.1. The diagram $\mathbf{LI}(G)$ of quotient pairs of G is the partially ordered set with objects $(G/K)_{G/L}$ for $L \subseteq K \subseteq G$, and with two types of morphisms. The *horizontal* morphisms (changing the x coordinate)

$$h_K^H : (G/K)_{G/L} \longrightarrow (G/H)_{G/L} \text{ for } L \subseteq K \subseteq H \subseteq G$$

and the *vertical* morphisms (changing the z coordinate)

$$v_L^K : (G/H)_{G/K} \longrightarrow (G/H)_{G/L} \text{ for } L \subseteq K \subseteq H \subseteq G.$$

In our principal example, the horizontal maps are localizations and the vertical maps are inflations, and this motivates the naming of the diagram.

We will refer to the terms $(G/H)_{G/L}$ with $\text{codim}(L) = \dim(G/L) = d$ as the *codimension d row*, and to those with $\text{codim}(H) = \dim(G/H) = d$ as the *codimension d column*. The terms $G/H = (G/H)_{G/1}$ form the *bottom row* and the terms $(G/L)_{G/L}$ form the *leading diagonal*.

By way of illustration, suppose G is of rank 2. The part of a diagram $R : \mathbf{LI}(G) \rightarrow \mathbb{C}$ including only one circle subgroup K would then take the form

$$\begin{array}{ccccccc}
 & & & & & & R(G/G)_{G/G} \\
 & & & & & & \downarrow \\
 & & & & R(G/K)_{G/K} & \longrightarrow & R(G/G)_{G/K} \\
 & & & & \downarrow & & \downarrow \\
 \begin{array}{c} \uparrow \\ z \end{array} & \begin{array}{c} \longrightarrow \\ x \end{array} & R(G/1)_{G/1} & \longrightarrow & R(G/K)_{G/1} & \longrightarrow & R(G/G)_{G/1}
 \end{array}$$

We think of the bottom row as consisting of the basic information, and the higher rows as adding minor refinements. The omission of brackets to write $R(G/K)_{G/L} = R((G/K)_{G/L})$ is convenient and follows conventions common in equivariant topology.

7.B. **The structure ring.** One particular diagram of rings is of special significance for us.

Definition 7.2. The algebraic structure diagram for G is the diagram of rings $R = \tilde{\mathcal{O}}_{\mathcal{F}/}$ defined by

$$R(G/K)_{G/L} := \mathcal{E}_{K/L}^{-1} \mathcal{O}_{\mathcal{F}/L}.$$

Noting that since $V^K = 0$ implies $V^H = 0$ we see that $\mathcal{E}_{H/L} \supseteq \mathcal{E}_{K/L}$, it is legitimate to take the horizontal maps to be localizations

$$h_K^H : \mathcal{E}_{K/L}^{-1} \mathcal{O}_{\mathcal{F}/L} \longrightarrow \mathcal{E}_{H/L}^{-1} \mathcal{O}_{\mathcal{F}/L}.$$

To define the vertical maps we begin with the inflation map $\text{inf}_{G/K}^{G/L} : \mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}/L}$, and then observe that if V is a representation of G/K with $V^H = 0$, it may be regarded as a representation of G/L , and Euler classes correspond in the sense that $\text{inf}(e_{G/K}(V)) = e_{G/L}(V)$. We therefore obtain a map

$$v_K^H : \mathcal{E}_{H/K}^{-1} \mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{E}_{H/L}^{-1} \mathcal{O}_{\mathcal{F}/L}.$$

Illustrating this for a group G of rank 2 in the usual way, we obtain

$$\begin{array}{ccccc} & & & & \mathcal{O}_{\mathcal{F}/G} \\ & & & & \downarrow \\ & & \mathcal{O}_{\mathcal{F}/K} & \longrightarrow & \mathcal{E}_{G/K}^{-1} \mathcal{O}_{\mathcal{F}/K} \\ & & \downarrow & & \downarrow \\ \begin{array}{c} \uparrow \\ z \\ \uparrow \\ \mathcal{O}_{\mathcal{F}} \end{array} & \xrightarrow{x} & \mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}_G^{-1} \mathcal{O}_{\mathcal{F}} \end{array}$$

At the top right, of course $\mathcal{O}_{\mathcal{F}/G} = \mathbb{Q}$, but it clarifies the formalism to use the more complicated notation.

7.C. **The category of R -modules.** In discussing modules, we need to refer to the structure maps for rings, so for a $\tilde{\mathcal{O}}_{\mathcal{F}/}$ -module M , if $L \subseteq K \subseteq H \subseteq G$, we generically write

$$\alpha_K^H : M(G/H)_{G/K} \longrightarrow M(G/H)_{G/L}$$

for the vertical map (i.e., changing the z coordinate), and

$$\tilde{\alpha}_K^H : \mathcal{E}_{H/L}^{-1} \mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} M(G/H)_{G/K} = (v_K^H)_* M(G/H)_{G/K} \longrightarrow M(G/H)_{G/L}$$

for the associated map of $\mathcal{O}_{\mathcal{F}/L}$ -modules. Similarly, we generically write

$$\beta_K^H : M(G/K)_{G/L} \longrightarrow M(G/H)_{G/L}$$

for the horizontal map (changing the x coordinate), which we refer to as the *basing map* after [15], and

$$\tilde{\beta}_K^H : \mathcal{E}_{H/L}^{-1} M(G/K)_{G/L} = (h_K^H)_* M(G/K)_{G/L} \longrightarrow M(G/H)_{G/L}$$

for the associated map of $\mathcal{E}_{H/L}^{-1} \mathcal{O}_{\mathcal{F}/L}$ -modules.

In our case the horizontal maps are simply localizations, so all maps in the G/L -row can reasonably be viewed as $\mathcal{O}_{\mathcal{F}/L}$ -module maps. On the other hand, the vertical maps increase the size of the rings, so it is convenient to replace the original diagram by the diagram in

which all vertical maps in the G/L column have had scalars extended to $\mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}}$. We refer to this as the $\tilde{\alpha}$ -diagram, and think of it as a diagram of $\mathcal{O}_{\mathcal{F}}$ -modules.

Definition 7.3. If M is an R -module, we say that M is *extended* if whenever $L \subseteq K \subseteq H$ the vertical map α_K^H is an extension of scalars along $v_K^H : \mathcal{E}_{H/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}$, which is to say that

$$\tilde{\alpha}_K^H : \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} M(G/H)_{G/K} \xrightarrow{\cong} M(G/H)_{G/L}$$

is an isomorphism of $\mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}$ -modules.

If M is an R -module, we say that M is *quasi-coherent* if whenever $L \subseteq K \subseteq H$ the horizontal map β_K^H is an extension of scalars along $h_K^H : \mathcal{E}_{K/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \longrightarrow \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}$, which is to say that

$$\tilde{\beta}_K^H : \mathcal{E}_{H/L}^{-1}M(G/K)_{G/L} \xrightarrow{\cong} M(G/H)_{G/L}$$

is an isomorphism.

We write qc- R -mod, e- R -mod and qce- R -mod for the full subcategories of R -modules with the indicated properties.

Next observe that the most significant part of the information in an extended object is displayed in its restriction to the leading diagonal. For example in our rank 2 example they take the form

$$\begin{array}{ccccc} & & & & M(G/G)_{G/G} \\ & & & & \downarrow \\ \begin{array}{c} \uparrow \\ z \\ \downarrow \\ \end{array} & \begin{array}{c} \xrightarrow{x} \end{array} & M(G/K)_{G/K} & \xrightarrow{\quad} & \mathcal{E}_{G/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \otimes_{\mathcal{O}_{\mathcal{F}/G}} M(G/G)_{G/G} \\ & & \downarrow & & \downarrow \\ M(G/1)_{G/1} & \xrightarrow{\quad} & \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} M(G/K)_{G/K} & \xrightarrow{\quad} & \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/G}} M(G/G)_{G/G} \end{array}$$

7.D. The category $\mathcal{A}(G)$ as a category of modules. We will typically abbreviate the previous diagram by just writing the final row. Thus, writing $\phi^K M = M(G/K)_{G/K}$, we obtain

$$\phi^1 M \longrightarrow \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M \longrightarrow \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/G}} \phi^G M,$$

leaving it implicit that the particular decomposition as a tensor product is part of the structure. This then corresponds to the notation for objects of $\mathcal{A}(G)$ in [18] and Section 2 above.

To make transition in the other direction explicit, we have a functor

$$\delta : \mathcal{A}(G) \longrightarrow \tilde{\mathcal{O}}_{\mathcal{F}/\text{-mod}}$$

defined by

$$\delta(M)(G/K)_{G/L} := \mathcal{E}_{K/L}^{-1}\phi^L M.$$

In particular, the two uses of the letter ϕ matches in the sense that

$$\phi^K \delta(M) = \phi^K M.$$

It is straightforward to encode the quasi-coherence condition of $\mathcal{A}(G)$ in the horizontal maps and the extendedness in the vertical maps.

Lemma 7.4. *The functor δ takes objects of $\mathcal{A}(G)$ to quasicohherent extended objects of $R\text{-mod}$. \square*

Part 3. Diagrams of G -spectra

8. DIAGRAMS AND FIXED POINTS

We will be assembling data about G -spectra from information about those with few isotropy groups. In this section we describe the basic process, which is conveniently done at the space level.

8.A. Universal spaces. If \mathcal{K} is a family of subgroups (i.e., a set of subgroups closed under conjugation and passage to smaller subgroups), there is a universal space $E\mathcal{K}$, characterized up to equivalence by the fact that $(E\mathcal{K})^H$ is empty if $H \notin \mathcal{K}$ and is contractible if $H \in \mathcal{K}$. We write $\tilde{E}\mathcal{K}$ for the unreduced suspension $S^0 * E\mathcal{K}$, so that there is a cofibre sequence

$$E\mathcal{K}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{K}.$$

It is sometimes convenient to use specific models for these universal spaces. For example, if V is an orthogonal representation, we may write $S(V)$ for the unit sphere and S^V for the one-point compactification of V . We then find

$$E\mathcal{K} = \bigcup_{k \geq 0} S(kV) \text{ and } \tilde{E}\mathcal{K} = \bigcup_{k \geq 0} S^{kV}$$

where $\mathcal{K} = \{H \mid V^H \neq 0\}$.

Finally, for normal subgroups H we define certain spaces $E\langle H \rangle$ by the cofibre sequence

$$E[\subset H]_+ \longrightarrow E[\subseteq H]_+ \longrightarrow E\langle H \rangle.$$

8.B. A convenient shorthand. Recall that the geometric fixed point functor extends the fixed point space functor in the sense that $\Phi^K \Sigma^\infty Y \simeq \Sigma^\infty(Y^K)$. The isotropy groups of a based G -space are the subgroups so that the fixed points are non-trivial, so it is natural to consider a stable, homotopy invariant version: the *geometric isotropy* of a G -spectrum X is defined by

$$\mathcal{GI}(X) := \{K \mid \Phi^K X \not\simeq *\}.$$

Certain G -spaces are useful in picking out isotropy information. The role of \mathcal{F} in $E\mathcal{F}_+$ and $\tilde{E}\mathcal{F}$ is complementary: the geometric isotropy of $E\mathcal{F}_+$ is \mathcal{F} and the geometric isotropy of $\tilde{E}\mathcal{F}$ is the complement of \mathcal{F} . It will help clarify arguments if we make the connection with geometric isotropy more direct: for certain collections \mathcal{N} of subgroups we will define a space or spectrum $X[\mathcal{N}]$ which captures the part of X with geometric isotropy in \mathcal{N} .

We are used to the idea of killing homotopy groups above some number to form a fibration

$$X[n, \infty) \longrightarrow X \longrightarrow X(-\infty, n-1].$$

Each of the maps has a suitable universal property embodied in the axioms of a t-structure.

Similarly, if \mathcal{C} is a cofamily of subgroups (i.e., a set of subgroups closed under conjugation and passage to larger subgroups) we may kill homotopy groups of geometric K -fixed points for K in the complement of \mathcal{C} and hence obtain a cofibration

$$X[\mathcal{C}^c] \longrightarrow X \longrightarrow X[\mathcal{C}].$$

Notice that the first map is a (non-equivariant) equivalence in Φ^K -fixed points for $K \in \mathcal{C}^c$ and the second map is a (non-equivariant) equivalence in Φ^K -fixed points for $K \in \mathcal{C}$. In the standard notation

$$X[\mathcal{C}^c] = X \wedge E(\mathcal{C}^c)_+ \text{ and } X[\mathcal{C}] = X \wedge \tilde{E}(\mathcal{C}^c).$$

The existence of the smash product and the fact that it commutes with geometric fixed points means that the cofibration can be obtained by smashing X with

$$E(\mathcal{C}^c)_+ \longrightarrow S^0 \longrightarrow \tilde{E}(\mathcal{C}^c).$$

This makes proofs of the results below very straightforward.

Of course Adams's notation for Postnikov systems is routinely extended to

$$X[m, n] = X(-\infty, n][m, \infty) \simeq X[m, \infty)(-\infty, n].$$

It is well understood that this is not functorial, but that the object is well defined up to equivalence.

Similarly for collections of subgroups.

Lemma 8.1. *If \mathcal{K} is a family of subgroups and \mathcal{C} is a cofamily then there is a weak equivalence*

$$X[\mathcal{C}][\mathcal{K}] \simeq X[\mathcal{K}][\mathcal{C}].$$

Furthermore, if \mathcal{K}' is another family and \mathcal{C}' is another cofamily with $\mathcal{C} \cap \mathcal{K} = \mathcal{C}' \cap \mathcal{K}'$ then

$$X[\mathcal{C}][\mathcal{K}] \simeq X[\mathcal{C}'][\mathcal{K}']. \quad \square$$

This allows us to define analogues of $X[m, n]$ for $m \leq n$.

Definition 8.2. We say that \mathcal{N} is a collection of subgroups has *no gaps* if it is closed under conjugation and if $L, H \in \mathcal{N}$ with $L \subseteq H$ then if $L \subseteq K \subseteq H$ then $K \in \mathcal{N}$. If \mathcal{N} has no gaps we write

$$X[\mathcal{N}] := X[\mathcal{K}][\mathcal{C}]$$

where \mathcal{K} and \mathcal{C} are chosen so that $\mathcal{N} = \mathcal{K} \cap \mathcal{C}$.

Remark 8.3. The point of this construction is that

$$\Phi^K(X[\mathcal{N}]) \simeq \begin{cases} \Phi^K X & \text{if } K \in \mathcal{N} \\ * & \text{if } K \notin \mathcal{N} \end{cases}$$

where the equivalences are non-equivariant. Thus $X[\mathcal{N}]$ is a version of X in which geometric isotropy outside \mathcal{N} has been killed.

Since S^0 is a unit for the smash product, we find the process is smashing in the sense that

$$X[\mathcal{N}] \simeq X \wedge S^0[\mathcal{N}].$$

Example 8.4. We now have new names for a number of familiar objects which clarify their roles:

$$S^0[\mathcal{K}] = EK_+, S^0[\mathcal{C}] = \tilde{E}\mathcal{C}^c, S^0[H] = E\langle H \rangle.$$

8.C. Geometric fixed points. It is clear that for spaces X the inclusion $X^H \rightarrow X$ induces an equivalence $X^H \wedge S^0[\supseteq H] \rightarrow X \wedge S^0[\supseteq H]$. When G is the torus there are particularly convenient models for $S^0[\not\supseteq H]$ and $S^0[\supseteq H]$.

Lemma 8.5. *When G is a torus, there are equivalences*

$$E[\not\supseteq H] = \bigcup_{V^H=0} S(V) \text{ and } \tilde{E}[\not\supseteq H] = \bigcup_{V^H=0} S^V. \quad \square$$

As a matter of notation, we write $\infty V(H) = \bigoplus_{V^H=0} V$, so that we have

$$S^0[\supseteq H] = \tilde{E}[\not\supseteq H] = S^{\infty V(H)},$$

and

$$X[\supseteq H] = X \wedge S^{\infty V(H)} \simeq \Phi^H X \wedge S^{\infty V(H)}$$

From this model it is easy to see that $S^{\infty V(H)}$ is a commutative ring up to homotopy.

8.D. Continuous limits. We want to take limits and homotopy limits over parts of the diagram $\mathbf{LI}(G)$. These are infinite, but not in a very aggressive way. The real essence of the diagram is contained in the part generated by any r circles L_1, L_2, \dots, L_r which are in general position in the sense that $G = L_1 \times \dots \times L_r$, which is to say the part containing the objects $(G/L_\sigma, a)$ where $\sigma \subseteq \{1, \dots, r\}$ and $L_\sigma = \prod_{i \in \sigma} L_i$.

In any case, we will be considering subdiagrams \mathbf{D} of $\mathbf{LI}(G)$, which are closed under passage to larger subgroups. We write $\mathcal{I}(\mathbf{D})$ for the set of objects initial in \mathbf{D} (i.e., without a proper predecessor in \mathbf{D}), and for any subset $\Sigma \subseteq \mathcal{I}(\mathbf{D})$ we write \mathbf{D}_Σ for the full diagram with objects

$$\text{ob}(\mathbf{D}_\Sigma) = \{\langle L_i \mid i \in \sigma \rangle \mid \sigma \subseteq \Sigma\}.$$

Definition 8.6. If $F : \mathbf{D} \rightarrow \mathbb{C}$, the *continuous inverse limit* of F over \mathbf{D} is

$$\lim'_{\leftarrow \mathbf{D}} F := \lim_{\rightarrow \Sigma} \lim_{\leftarrow \mathbf{D}_\Sigma} F,$$

where Σ runs through the finite subsets of $\mathcal{I}(\mathbf{D})$. The definition of continuous homotopy inverse limits is precisely similar.

Smash products and geometric fixed points do not behave well for homotopy inverse limits, but because they commute with sums, and finite products are sums, they do behave well for continuous limits.

Lemma 8.7. *Smash products and geometric fixed points commute with continuous homotopy inverse limits.* \square

8.E. Continuous limits and universal spaces. The following elementary space level result is a basic input.

Lemma 8.8. *The continuous homotopy pullback of the diagram $S^{\infty V(\bullet)}$ restricted to the diagram $\mathbf{NTConnSub}(G)$ of non-trivial connected subgroups is $\tilde{E}\mathcal{F}$:*

$$\text{holim}'_{\leftarrow H \in \mathbf{NTConnSub}(G)} S^{\infty V(H)} \simeq \tilde{E}\mathcal{F}.$$

Proof: We verify this at the space level by considering fixed points.

The continuous limit is the direct limit over finite initial sets Σ , which is to say over finite collections L_1, \dots, L_s of circles. If $\sigma \subseteq \{1, \dots, s\}$ we write

$$L_\sigma = \langle L_i \mid i \in \sigma \rangle$$

for the subgroup generated by the corresponding subgroups. Now choose a closed subgroup K and consider geometric K -fixed points; as usual $(S^{\infty V(L_\sigma)})^K$ is S^0 if $L_\sigma \subseteq K$ and is contractible otherwise. Evidently the fixed point set is S^0 precisely when σ is a non-empty subset of

$$\Sigma(K) := \{i \mid L_i \subseteq K\}.$$

In other words, we have a punctured cube with all vertices S^0 , and by the inclusion-exclusion formula the homotopy limit is thus S^0 if $\Sigma(K) \neq \emptyset$ and contractible otherwise. It is clear that as Σ increases in size, so does $\Sigma(K)$, so the maps of homotopy inverse limits are all equivalences once $\Sigma(K) \neq \emptyset$. Since the finite subgroups are precisely those not containing a circle, the result follows. \square

9. HASSE DIAGRAMS OF RING G -SPECTRA

9.A. Diagrams of ring spectra. We are going to begin by describing a diagram of ring spectra modelled on

$$\mathbf{LC}(G) = \mathbf{ConnQuot}(\mathbf{G}) \times \{n \longrightarrow c\}.$$

The most basic diagram of rings shaped on $\mathbf{ConnSub}(\mathbf{G})$ is

$$S^{\infty V(\bullet)} : \mathbf{ConnSub}(\mathbf{G}) \longrightarrow G\text{-spectra}$$

defined by

$$S^{\infty V(\bullet)}(H) := S^{\infty V(H)}.$$

For consistency with usage elsewhere we identify $\mathbf{ConnSub}(\mathbf{G})$ with $\mathbf{ConnQuot}(\mathbf{G})$ by taking each connected subgroup K to the quotient group G/K as usual.

In the case when G is a circle, this diagram $S^0 \longrightarrow S^{\infty V(G)}$ is the top (or ‘natural’) row of the Hasse square. We need to provide the counterpart of the second (or ‘complete’) row of the Hasse square as well.

Definition 9.1. The *isotropic Hasse diagram* for the ring G -spectrum S^0 is the $\mathbf{LC}(G)$ -diagram of ring G -spectra obtained by smashing $S^{\infty V(\bullet)}$ with $S^0 \longrightarrow DEF_+$, and denoted \tilde{R}_{top} . More precisely,

$$\tilde{R}_{top}(G/H, n) = S^{\infty V(H)} \text{ and } \tilde{R}_{top}(G/H, c) = S^{\infty V(H)} \wedge DEF_+.$$

As before, we must comment that the n values are only ring spectra up to homotopy.

Just as we omitted the top left entry for a circle in Subsection 4.D we will show that the diagram can be reconstructed from a much smaller subdiagram in general.

Example 9.2. For example if $r = 2$ the diagram \tilde{R}_{top} is

$$\begin{array}{ccccccc}
 & & S^0 & \longrightarrow & S^{\infty V(H)} & \longrightarrow & S^{\infty V(G)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 y \uparrow & & DEF_+ & \longrightarrow & S^{\infty V(H)} \wedge DEF_+ & \longrightarrow & S^{\infty V(G)} \wedge DEF_+ \\
 & \xrightarrow{x} & & & & &
 \end{array}$$

(where the middle vertical is a representative for the countably many circle subgroups H of G) and this can be reconstructed even if we omit S^0 and $S^{\infty V(H)}$ for all circles H .

9.B. Model structures on diagrams. Now consider the $\mathbf{LC}(G)$ -diagrams of modules, in the light of the contents of Subsection 5.B and Theorem B.1.

In all cases with $a = c$, $\tilde{R}_{top}(G/K, c)$ is a commutative ring spectrum and we use the usual projective model structure on $\tilde{R}_{top}(G/K, c)$ -modules. The localization maps $\tilde{R}_{top}(G/L, c) \rightarrow \tilde{R}_{top}(G/K, c)$ for $L \subseteq K$ induce the Quillen pair on module categories given by restriction and extension of scalars.

In all cases with $a = n$, the category of $\tilde{R}_{top}(G/K, n)$ -modules is by definition the category of G -spectra localized to invert maps which become equivalences when smashed with $S^{\infty V(K)}$. All the localization maps are left Quillen functors.

Finally the completion functors associated to the map $\tilde{R}_{top}(G/L, n) \rightarrow \tilde{R}_{top}(G/L, c)$ is a composite of two left Quillen functors

$$G\text{-spectra}/L \rightarrow DEF_+\text{-module-}G\text{-spectra}/L \rightarrow S^{\infty V(K)} \wedge DEF_+\text{-module-}G\text{-spectra}.$$

The first is the localization of the extension of scalars along $\mathbb{S} \rightarrow DEF_+$ and the second (which is a Quillen equivalence) is induced from extension of scalars along $DEF_+ \rightarrow S^{\infty V(K)} \wedge DEF_+$, which exists since $S^{\infty V(K)} \wedge DEF_+$ -modules are local.

9.C. Extensions of scalars. We have defined the isotropic Hasse diagram $\tilde{R}_{top}^{\mathbf{LC}(G)} := \tilde{R}_{top}$, which is a diagram of G -spectra modelled on $\mathbf{LC}(G)$. We would like to incorporate the information from the quotients of G by connected subgroups in a larger diagram modelled on $\mathbf{LCI}(G)$.

The first obstacle to this is that $\tilde{R}_{top}^{\mathbf{LC}(G/K)}$ is a diagram of G/K -spectra, so we need to pause to explain how the adjunction between inflation and Lewis-May fixed points allows us to make sense of this. Once again we use the formalism explained in Subsection 5.B. Indeed, we must specify a (vertical, inflation) morphism $v_L^K : (G/H, a)_{G/K} \rightarrow (G/H, a)_{G/L}$ with $L \subseteq K \subseteq H \subseteq G$.

To start with, there is a Lewis-May K/L -fixed point functor

$$(\cdot)^{K/L} : G/L\text{-spectra} \rightarrow G/K\text{-spectra}$$

with inflation as a left adjoint. If we start with the ring G/L -spectrum $\tilde{R}_{top}(G/H, a)_{G/L}$ we shall see in Section 11 that the fixed point functor extends to a functor

$$\Psi^{K/L} : \tilde{R}_{top}(G/H, a)_{G/L}\text{-module-}G/L\text{-spectra} \rightarrow [\tilde{R}_{top}(G/H, a)_{G/L}]^{G/K}\text{-module-}G/K\text{-spectra}$$

with inflation and extension of scalars giving a left adjoint. Thus, provided we can construct a ring map $\tilde{R}_{top}(G/H, a)_{G/K} \longrightarrow [\tilde{R}_{top}(G/H, a)_{G/L}]^{G/K}$, we obtain a left Quillen functor

$$\mathbb{M}(G/H, a)_{G/K} = \tilde{R}_{top}(G/H, a)_{G/K}\text{-mod-}G/K\text{-spectra} \longrightarrow \\ \tilde{R}_{top}(G/H, a)_{G/L}\text{-mod-}G/L\text{-spectra} = \mathbb{M}(G/K, a)_{G/L}$$

which is the appropriate input to the model structures on \mathbb{M} -diagram categories described in Theorem B.1. It remains to construct the appropriate maps of rings (which is to say that in the case $a = n$ we construct the corresponding map of model categories).

On the $a = n$ side of the diagram, we note that when $L \subseteq K \subseteq H \subseteq G$ there is a map

$$\tilde{R}_{top}(G/H, n)_{G/K} = \inf \left(\bigcup_{V^{H/K}=0} S^V \right) \longrightarrow \left(\bigcup_{W^{H/L}=0} S^W \right) = \tilde{R}_{top}(G/H, n)_{G/L}$$

where V runs through representations of G/K and W runs through the (larger) set of representations of G/L . If the rings in question had strictly commutative models we would combine extension of scalars with inflation, but instead we simply use the inflation functor from G/K -spectra to G -spectra, and then localize to form the categories of spectra over H .

Turning the $a = c$ part of the diagram, we operate with a map of strictly commutative ring spectra. Recall that \mathcal{F} denotes the family of finite subgroups of G and that \mathcal{F}/K denotes the family of finite subgroups of G/K . We shall see in Section 11 that there is a map of ring G/K -spectra

$$D(E\mathcal{F}/K_+) \longrightarrow (DE\mathcal{F}_+)^K,$$

and an adjunct map

$$\inf D(E\mathcal{F}/K_+) \longrightarrow DE\mathcal{F}_+$$

of ring G -spectra.

Combining all this, we may form the diagram $\tilde{R}_{top}^{\mathbf{LCI}(G)}$ of ring G -spectra modelled on $\mathbf{LCI}(G)$ by combining the diagrams $\tilde{R}_{top}^{\mathbf{LC}(G/K)}$, and by Theorem B.1 we may put projective and injective models on the category of diagrams of modules. Note that for $\mathbf{LI}(G)$ -diagrams, all the rings are strictly commutative ring spectra, so each individual model category is an actual category of module spectra rather than a localization chosen to fulfil the role.

Definition 9.3. The diagram $\tilde{R}_{top}^{\mathbf{LCI}(G)}$ is defined by

$$\tilde{R}_{top}^{\mathbf{LCI}(G)}(G/K, a)_{G/L} := \inf \left[\tilde{R}_{top}^{\mathbf{LC}(G/L)}(\overline{G}/\overline{K}, a) \right]$$

where $L \subseteq K \subseteq G$ and bars denote quotients by L . The horizontal maps and completion maps are inflations of those for the components $R_{G/L}$. The vertical maps are the maps just described.

Illustrating the $\mathbf{LI}(G)$ -diagram of rings for a group G of rank 2 in the usual way, we obtain

$$\begin{array}{ccccc}
 & & & \inf DEF/G_+ & \\
 & & & \downarrow & \\
 \begin{array}{c} \uparrow z \\ \xrightarrow{x} \end{array} & \inf DEF/K_+ & \longrightarrow & S^{\infty V(G/K)} \wedge \inf DEF/K_+ & \\
 & \downarrow & & \downarrow & \\
 DEF_+ & \longrightarrow & S^{\infty V(K)} \wedge DEF_+ & \longrightarrow & S^{\infty V(G)} \wedge DEF_+
 \end{array}$$

At the top right, of course $EF/G_+ = S^0$, but clarifies the formalism to use the more complicated notation.

The following statement is that the resulting diagram is extended in the analogous sense to that in Subsection 7.C, and that therefore the whole diagram $\tilde{R}_{top}^{\mathbf{LCI}(G)}$ contains little more information than the diagram $\tilde{R}_{top}^{\mathbf{LC}(G)}$. This is already rather obvious for the n face, where geometric fixed points explicitly recover the information for subquotients. Accordingly we can concentrate on the c face. We prove the result with an extra parameter spectrum X , to emphasize the link with Section 13, but with $X = S^0$ the result simply states that extension of scalars from each row, in the $\mathbf{LI}(G)$ diagram for \tilde{R}_{top} gives the corresponding part of the next row.

Lemma 9.4. *There is an equivalence*

$$S^{\infty V(K)} \wedge DEF_+ \wedge X \simeq (S^{\infty V(K)} \wedge DEF_+) \otimes_{\inf DEF/K_+} \inf(DEF/K_+ \wedge \Phi^K X)$$

natural in X .

Proof: We take the map $DEF/K_+ \rightarrow (DEF_+)^K$, and smash with $\Phi^K X = (S^{\infty V(K)} \wedge X)^K$ to obtain

$$DEF/K_+ \wedge \Phi^K X \rightarrow (DEF_+)^K \wedge (S^{\infty V(K)} \wedge X)^K.$$

Using the fact that taking Lewis-May fixed points is lax monoidal, we obtain a map

$$DEF/K_+ \wedge \Phi^K X \rightarrow (DEF_+ \wedge S^{\infty V(K)} \wedge X)^K.$$

Taking adjoints and using the universal property of extension of scalars, we obtain a natural transformation of the indicated functors.

To see this is an equivalence of functors we note that the right hand side is

$$(S^{\infty V(K)} \wedge DEF_+) \otimes_{\inf DEF/K_+} \inf(DEF/K_+ \wedge \inf(\Phi^K X)) \simeq (S^{\infty V(K)} \wedge DEF_+) \wedge \inf(\Phi^K X),$$

so this amounts to smashing the usual equivalence

$$S^{\infty V(K)} \wedge X \simeq S^{\infty V(K)} \wedge \inf(\Phi^K X)$$

with DEF_+ . □

10. FILLING IN THE BLANKS

In this section we show that the category of G -spectra is equivalent to the cellularization of a category of modules over a diagram of ring G -spectra.

10.A. **The theorem.** We have now described an $\mathbf{LCI}(G)$ -diagram \tilde{R}_{top} of equivariant spectra. The purpose of this section is to establish the main diagram-theoretic Quillen equivalence.

Theorem 10.1. *There is a Quillen equivalence*

$$G\text{-spectra} \xrightarrow{\simeq} \text{cell-}\tilde{R}_{top}^{\mathbf{LI}}\text{-mod-}G\text{-spectra},$$

where cellularization is with respect to the image of the cells G/H_+ .

The idea is that each side consists of modules over a diagram of ring G -spectra, with the usual interpretation via localizations for the part of the diagram indexed with n . Each of the two diagrams is a restriction of the single diagram $\mathbf{LCI}(G)$, and the two diagrams of rings are restrictions of a single $\mathbf{LCI}(G)$ -diagram \tilde{R}_{top} of rings (on the left, the category of G -spectra comes from the diagram with the single object $(G/1, n)_{G/1}$ and on the right, the diagram is $\mathbf{LI}(G)$). We will show that the two subdiagrams are each cellularly dense in $\mathbf{LCI}(G)$ in the sense of Section 5.

In fact, each of the cellularly dense inclusions is the composite of about $2r$ elementary cellularly dense inclusions, alternating left and right. Altogether, this means the equivalence in the theorem is a composite of about $8r$ simpler equivalences, each of which arises from either a right cellularly dense inclusion (essentially an extension of scalars), a left cellularly dense inclusion (essentially a pullback) or a change of models. We outlined this in Subsection 5.F, and we now provide more details.

Indeed, we decomposed the inclusions into elementary inclusions of four types, Type $\mathbf{LC1}$, Type $\mathbf{LC2}$, Type $\mathbf{LI1}$ and Type $\mathbf{LI2}$. We will treat these four cases in turn. As mentioned before, the last one (in which the diagram is filled in by a pullback) is the one with real content.

Remark 10.2. The general proof of Theorem 10.1 given here may of course be applied to the case when G is the circle group and $r = 1$. The resulting equivalence involves more steps than that given in Section 4. First, since the only non-trivial connected subgroup is G , we may restrict to the diagram $\mathbf{LC}(G)$ inside $\mathbf{LCI}(G)$. With this replacement, $\mathbf{LI}(G)$ is the Hasse fork. With this simplification, the present method involves comparisons

$$\left(\begin{array}{ccc} (G/1, n) & \text{---} & \\ \vdots & & \vdots \\ \text{---} & & \text{---} \end{array} \right) \rightarrow \left(\begin{array}{ccc} (G/1, n) & \longrightarrow & (G/G, n) \\ \downarrow & & \downarrow \\ (G/1, c) & \longrightarrow & (G/G, c) \end{array} \right) \leftarrow \left(\begin{array}{ccc} & \text{---} & (G/G, n) \\ \vdots & & \downarrow \\ (G/1, c) & \longrightarrow & (G/G, c) \end{array} \right)$$

In Section 4 the left and right diagrams were related directly.

10.B. **The cells.** To make sense of the strategy, we need to introduce the sets of cells. The cells are precisely those obtained from the orbits G/H_+ in the category of G -spectra where H runs through the closed subgroups of G . The easiest way to explain this is to say that the cells in the $\mathbf{LC}(G)$ -diagram category are the objects of the form $\tilde{R}_{top} \wedge G/H_+$, and for each layer $\mathbf{LC}(G/K)$ we take $\tilde{R}_{top} \wedge \Phi^K(G/H_+)$. For any of the subdiagrams \mathbf{D} that we consider, we simply restrict these modules to the subdiagram.

To see that the context is correct we need to see that for each elementary inclusion $i : \mathbf{D} \rightarrow \mathbf{E}$ of subdiagrams, the cells over \mathbf{E} come from \mathbf{D} in the sense of Subsection 5.D.

For the right cellularly dense inclusions we use the adjunction (i_*, i^*) and we consider the counit $i_* i^* \sigma \rightarrow \sigma$. In effect we are considering a map $R_1 \rightarrow R_2$ and showing that the natural map

$$R_2 \otimes_{R_1} (R_1 \wedge G/H_+) \rightarrow R_2 \wedge G/H_+$$

is an equivalence. In this toy example it is obvious, and this is essentially the general case; we explain this more fully for **LC1** and **LI1** in the appropriate places below.

For the left cellularly dense inclusions we use the adjunction $(i^*, i_!)$ and we consider the counit $\sigma \rightarrow i_! i^* \sigma$. In effect, σ will be a diagram obtained from one of the form

$$\begin{array}{ccc} R_1 & \longrightarrow & R_2 \\ \downarrow & & \downarrow \\ R_3 & \longrightarrow & R_4 \end{array}$$

by smashing with G/H_+ . The restriction $i^* \sigma$ is obtained by omitting the top left entry, and then $i_! i^* \sigma$ is obtained by filling in using the pullback. Thus the condition states that the square is a homotopy pullback. Since smashing with G/H_+ preserves pullbacks, it suffices to check the case when $G/H_+ = S^0$. This is in effect what was originally motivated in Subsection 4.D and done in detail in Section 8. We comment further on the actual examples in the subsections on **LC2** and **LI2** below.

10.C. Type LC1. We recall that for each $s \geq 0$, $\mathbf{LC}_s^*(G)$ consists of all subdiagrams $\mathbf{LC}(G/K)$ for $\dim(K) < s$ together with $(G/H, n)_{G/H}$ for $\dim(H) = s$, and $\mathbf{LC}_s(G)$ consists of all subdiagrams $\mathbf{LC}(G/K)$ for $\dim(K) \leq s$.

This gives the inclusion

$$\mathbf{LC}_s^*(G) \subseteq \mathbf{LC}_s(G),$$

which we may illustrate when the rank is 2 and $s = 1$ in the following diagram

$$\begin{array}{ccccccc} & & & & + & \xrightarrow{\hspace{2cm}} & + \\ & & & \nearrow & \downarrow & & \downarrow \\ & & (G/H, n)_{G/H} & \xrightarrow{\hspace{1cm}} & + & \nearrow & + \\ & & \downarrow & & \downarrow & & \downarrow \\ (G/1, c)_{G/1} & \xrightarrow{\hspace{1cm}} & (G/H, c)_{G/1} & \xrightarrow{\hspace{1cm}} & (G/G, c)_{G/1} & & \\ \nearrow & & \nearrow & & \nearrow & & \\ (G/1, n)_{G/1} & \xrightarrow{\hspace{1cm}} & (G/H, n)_{G/1} & \xrightarrow{\hspace{1cm}} & (G/G, n)_{G/1} & & \end{array}$$

where the points marked $+$ are in $\mathbf{LC}_s(G)$ but not $\mathbf{LC}_s^*(G)$.

For brevity we write RC_s^* for the diagram of rings over $\mathbf{LC}_s^*(G)$ and RC_s for the diagram of rings over $\mathbf{LC}_s(G)$. Thus RC_s^* is the restriction of RC_s , and there is a corresponding restriction map

$$\text{res} : RC_s\text{-mod} \rightarrow RC_s^*\text{-mod}.$$

The left adjoint takes an RC_s^* -module M to an RC_s -module $RC_s \otimes_{R_s^*} M$ defined as follows. On objects $(G/H, a)_{G/K}$ with $\dim(K) < s$ there is no change, and if $\dim(K) = s$ we note that there is an initial object amongst all those of the form $(G/H, a)_{G/K}$: indeed, there is a morphism

$$(G/K, n)_{G/K} \rightarrow (G/H, a)_{G/K}$$

in $\mathbf{LC}(G/K)$. Accordingly, we may take

$$(RC_s \otimes_{RC_s^*} M)(G/H, a)_{G/K} := RC_s(G/H, a)_{G/K} \otimes_{RC_s(G/K, n)_{G/K}} M(G/K, n)_{G/K}.$$

This immediately supplies all the morphisms between objects of $\mathbf{LC}(G/K)$. We also need to consider inflation maps, so we consider groups $L \subseteq K \subseteq H$ and the associated vertical map $(G/H, a)_{G/K} \longrightarrow (G/H, a)_{G/L}$. For this we consider the square

$$\begin{array}{ccc} M(G/K, n)_{G/K} & \longrightarrow & RC_s(G/H, a)_{G/K} \otimes_{RC_s(G/K, n)_{G/K}} M(G/K, n)_{G/K} \\ \downarrow & & \downarrow \\ M(G/K, n)_{G/L} & \longrightarrow & M(G/H, a)_{G/L} \end{array}$$

allowing us to define the vertical map by the universal property of extension of scalars.

Lemma 10.3. *There is a Quillen adjunction on the diagram-projective model structures.*

$$RC_s \otimes_{RC_s^*} (\cdot) : RC_s^* \text{-mod} \rightleftarrows RC_s \text{-mod} : \text{res}$$

This induces a Quillen equivalence on the associated cellularizations.

$$\text{cell-}RC_s^* \text{-mod} \xrightarrow{\simeq} \text{cell-}RC_s \text{-mod}$$

This is a Quillen adjunction; res preserves weak equivalences and fibrations since they are defined objectwise in the diagram-projective model structures.

Finally, for cells the derived counit is the identity at all points of the smaller diagram $\mathbf{LC}_s^*(G)$, and at each point of the larger diagram it is exactly as in the toy model of Subsection 10.B. Indeed, to deal with the point $(G/H, a)_{G/K}$ with $\dim(K) = s$ we take $R_1 = RC_s(G/K, n)_{G/K}$ and $R_2 = RC_s(G/H, a)_{G/K}$.

10.D. **Type LC2.** For each $s \geq 0$, we recall that $\mathbf{LC}_{s+1}^*(G)$ is obtained from $\mathbf{LC}_s(G)$ by adjoining the objects $(G/K, n)_{G/K}$ with $\dim(K) = s + 1$. This gives the inclusion

$$\mathbf{LC}_s(G) \subseteq \mathbf{LC}_{s+1}^*(G),$$

which we may illustrate when the rank is 2 and $s = 0$ in the following diagram

where the point marked $*$ is in $\mathbf{LC}_{s+1}^*(G)$ but not $\mathbf{LC}_s(G)$.

Continuing with the notations RC_s^* and RC_{s+1}^* for the diagrams of rings over $\mathbf{LC}_s(G)$ and $\mathbf{LC}_{s+1}^*(G)$, there is a corresponding restriction map

$$\text{res} : RC_{s+1}^* \text{-mod} \longrightarrow RC_s \text{-mod}.$$

This time we need additional structure to supply an adjoint, and in our application, res is a left adjoint, with the right adjoint being given by a Lewis-May fixed point functor. Indeed, the right adjoint takes an RC_s -module M to an RC_{s+1}^* -module $\Psi_s M$ defined as follows. On objects of $\mathbf{LC}_s(G)$ there is no change, and if $\dim(K) = s + 1$ we take

$$(\Psi_s M)(G/K, n)_{G/K} := \lim_{\leftarrow L} \Psi^{K/L} [M(G/K, n)_{G/L}].$$

The morphisms between objects of $\mathbf{LC}_s(G)$ are just as before. For the inflation maps we use the unit of the inflation fixed point adjunction.

Lemma 10.4. *There is a Quillen adjunction of the diagram-injective model structures.*

$$\text{res} : R_s\text{-mod} \rightleftarrows R_{s+1}^*\text{-mod} : \Psi_s$$

This induces a Quillen equivalence on the associated cellularizations.

$$\text{cell-}R_s\text{-mod} \xrightarrow{\simeq} \text{cell-}R_{s+1}^*\text{-mod}$$

Proof: This is a Quillen adjunction; res preserves weak equivalences and cofibrations since they are defined objectwise in the diagram-injective model structures.

Finally, we consider cells. The point here is that for any extended module, the limit diagram is constant. More specifically, the cells are $RC_s \wedge X$ with $X = G/J_+$ for some closed subgroup J and

$$RC_s(G/K, n)_{G/L} = S^{\infty V(K/L)}$$

and therefore

$$\Psi^{K/L}(RC_s(G/K, n)_{G/L}) = \Phi^{K/L} \Phi^L X = \Phi^K X,$$

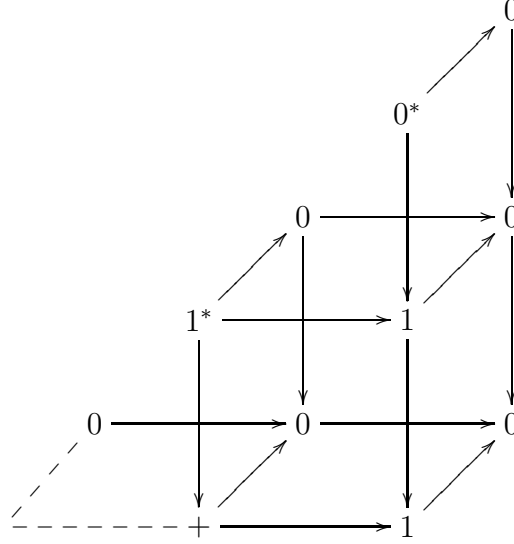
independent of L . Accordingly, $\Psi_s(RC_s \wedge X)(G/K, n)_{G/K}$, the limit of the constant diagram, is $\Phi^K X$ as required. \square

10.E. Type LI1. We recall that for each $s \geq 0$, $\mathbf{LI}_*^s(G)$ consists of all complete objects $(G/H, c)_{G/K}$ together with objects $(G/H, n)_{G/K}$ for $\text{codim}(H) < s$ and the objects $(G/H, n)_{G/H}$ with $\text{codim}(H) = s$. The diagram $\mathbf{LI}^{s+1}(G)$ is obtained from $\mathbf{LI}_*^s(G)$ by adjoining all objects $(G/H, n)_{G/K}$ with $\text{codim}(H) = s$ and $\text{codim}(K) > s$.

This gives the inclusion

$$\mathbf{LI}_*^s(G) \subseteq \mathbf{LI}^{s+1}(G),$$

which we may illustrate when the rank is 2 and $s = 1$ in the following diagram



where the points marked $+$ are in $\mathbf{L I}^{s+1}(G)$ but not $\mathbf{L I}_*^s(G)$.

For brevity we write RI_*^s for the diagram of rings over $\mathbf{L I}_*^s(G)$ and RI^{s+1} for the diagram of rings over $\mathbf{L I}^{s+1}(G)$. Thus RI_*^s is the restriction of RI^{s+1} , and there is a corresponding restriction map

$$\text{res} : RI^{s+1}\text{-mod} \longrightarrow RI_*^s\text{-mod}.$$

The left adjoint takes an RI_*^s -module M to an RI^{s+1} -module $RI^{s+1} \otimes_{RI_*^s} M$ defined as follows. On complete objects, on objects $(G/H, n)_{G/K}$ with $\text{codim}(H) < s$ and on objects $(G/H, n)_{G/H}$ there is no change. On an object $(G/H, n)_{G/K}$ with $\text{codim}(H) = s$ and $\text{codim}(K) > s$, we take

$$(RI^{s+1} \otimes_{RI_*^s} M)(G/H, n)_{G/K} := RI^{s+1}(G/H, n)_{G/K} \otimes_{RI_*^s(G/H, n)_{G/H}} M(G/H, n)_{G/H}.$$

This immediately supplies all vertical morphisms between objects. We also need to consider completion maps. For this we consider the square

$$\begin{array}{ccc} M(G/H, n)_{G/H} & \xrightarrow{\quad} & M(G/H, c)_{G/H} \\ \downarrow & & \downarrow \\ R_s(G/H, a)_{G/K} \otimes_{R_s(G/K, n)_{G/K}} M(G/K, n)_{G/K} & \longrightarrow & M(G/H, c)_{G/K} \end{array}$$

allowing us to define the lower horizontal map by the universal property of extension of scalars.

Lemma 10.5. *There is a Quillen adjunction on the diagram-projective model structures.*

$$RI^{s+1} \otimes_{RI_*^s} (\cdot) : RI_*^s\text{-mod} \rightleftarrows RI^{s+1}\text{-mod} : \text{res}$$

This induces a Quillen equivalence on the associated cellularizations.

$$\text{cell-}RI_*^s\text{-mod} \xrightarrow{\simeq} \text{cell-}RI^{s+1}\text{-mod}$$

Proof: This is a Quillen adjunction; res preserves weak equivalences and fibrations since they are defined objectwise in the diagram-projective model structures.

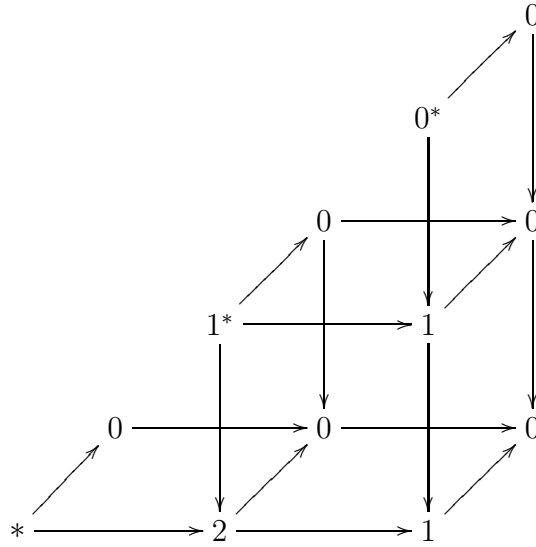
Finally, for cells the counit is the identity at all points of the smaller diagram $\mathbf{LI}_*^s(G)$, and at each point of the larger diagram it is exactly as in the toy model of Subsection 10.B. Indeed, to deal with the point $(G/H, n)_{G/K}$ with $\text{codim}(H) = s$ we take $R_1 = RI_*^s(G/H, n)_{G/H}$ and $R_2 = RI^{s+1}(G/H, n)_{G/K}$. \square

10.F. **Type LI2.** We recall that for each $s \geq 0$, $\mathbf{LI}_*^s(G)$ consists of all complete objects $(G/H, c)_{G/K}$ together with objects $(G/H, n)_{G/K}$ for $\text{codim}(H) < s$ and the objects $(G/H, n)_{G/H}$ with $\text{codim}(H) = s$. The diagram $\mathbf{LI}^s(G)$ is obtained from $\mathbf{LI}_*^s(G)$ by omitting all the objects $(G/H, n)_{G/H}$ with $\text{codim}(H) = s$.

This gives the inclusion

$$\mathbf{LI}^s(G) \subseteq \mathbf{LI}_*^s(G),$$

which we may illustrate when the rank is 2 and $s = 2$ in the following diagram



where the point marked $*$ is in $\mathbf{LI}_*^s(G)$ but not $\mathbf{LI}^s(G)$.

We would like to fill in the additional objects by pullbacks, but this gives the wrong answer. Instead we use continuous pullbacks as described in Subsection 8.D. However, to obtain adjoint pairs we need to factorize the functor into three.

To explain this we need to introduce two intermediate categories. To start with, we consider various subdiagrams $\mathbf{LI}_F^s(G)$. The indexing set F consists of a finite set of circles L_1, L_2, \dots, L_n in G . This gives rise to the set of F -generated subgroups, namely the subgroups $L_A := \langle L_a \mid a \in A \rangle$ generated by the circles L_a as a runs through the subset A of F . More generally, the images in G/K of these F -generated subgroups of G will also be called F -generated. The diagram $\mathbf{LI}_F^s(G)$ contains all complete objects $(G/H, c)_{G/K}$ and all objects $(G/H, n)_{G/K}$ in $\mathbf{LI}^s(G)$ where H/K is F -generated. The diagram $\mathbf{LI}_{*F}^s(G)$ is obtained from $\mathbf{LI}_F^s(G)$ by adjoining the objects $(G/H, n)_{G/H}$ with $\text{codim}(H) = s$.

Next we consider the category $RI_F^s\text{-mod}$ of modules over the restriction to $\mathbf{LI}_F^s(G)$. Note that if $F' \subseteq F$, there is a restriction functor

$$RI_F^s\text{-mod} \longrightarrow RI_{F'}^s\text{-mod}.$$

Accordingly, we may consider the diagram category $RI_{\mathcal{F}}^s\text{-mod}$ whose objects are systems $\{M_F\}_F$, where M_F is an RI_F^s -module, and if $F' \subseteq F$ we have a map $M_{F'} \longrightarrow M_F$ of $RI_{F'}^s$ -modules.

The diagram \widehat{RI}_F^s is an $\mathbf{LI}_{*F}^s(G)$ -diagram of rings, but it is not the restriction of R . Instead, it agrees with RI_F^s over $\mathbf{LI}_F^s(G)$, and the remaining points are filled in by taking limits:

$$\widehat{RI}_F^s(G/K, n)_{G/K} := \lim_{\leftarrow F\text{-generated } H/K} R(G/H, n)_{G/K}.$$

We may then consider the module category $\widehat{RI}_F^s\text{-mod}$. Again we note that if $F' \subseteq F$, there is a restriction functor

$$\widehat{RI}_F^s\text{-mod} \longrightarrow \widehat{RI}_{F'}^s\text{-mod}.$$

Accordingly, we may consider the diagram category $\widehat{RI}_{\mathcal{F}}^s\text{-mod}$ whose objects are systems $\{M_{*F}\}_F$, where M_{*F} is an \widehat{RI}_F^s -module, and if $F' \subseteq F$ we have a map $M_{*F'} \longrightarrow M_{*F}$ of $\widehat{RI}_{F'}^s$ -modules.

This introduces the two intermediate categories $RI_{\mathcal{F}}^s\text{-mod}$ and $\widehat{RI}_{\mathcal{F}}^s\text{-mod}$, and we now turn to constructing adjunctions relating them. Just as we obtained \widehat{RI}_F^s by filling in RI_F^s using finite limits, we may do the same for modules. This gives an adjunction

$$res : \widehat{RI}_F^s\text{-mod} \xrightleftharpoons{\quad} RI_F^s\text{-mod} : \lim_{\leftarrow}.$$

As F varies, these adjunctions are compatible and fit together to give an adjunction

$$res^{\mathcal{F}} : \widehat{RI}_{\mathcal{F}}^s\text{-mod} \xrightleftharpoons{\quad} RI_{\mathcal{F}}^s\text{-mod} : \lim_{\leftarrow}^{\mathcal{F}}.$$

We may now explain that the two new intermediate categories appear in the sequence

$$RI^s\text{-mod} \xrightleftharpoons[\lim_{\leftarrow}]{res_1} RI_{\mathcal{F}}^s\text{-mod} \xrightleftharpoons[\lim_{\leftarrow}^{\mathcal{F}}]{res^{\mathcal{F}}} \widehat{RI}_{\mathcal{F}}^s\text{-mod} \xrightleftharpoons[\lim_{\leftarrow}^{\mathcal{F}}]{\lim_{\rightarrow F}} RI_{*}^s\text{-mod}$$

of adjunctions (with left adjoints on top as usual).

The first adjunction is the standard adjunction given by restriction and pullback. The central adjunction has been described. For the third adjunction, note that by the universal property of the inverse limit, there is a map $RI_{*F}^s \longrightarrow \widehat{RI}_F^s$ of RI_{*F}^s -diagrams of rings. This is compatible with the restriction maps arising from $F' \subseteq F$, and therefore gives a restriction functor res_3 as indicated.

As for the model structures, each of the three are Quillen adjunctions when we use the appropriate models. The right hand adjunction requires projective model structures so that restriction preserves the objectwise weak equivalences and fibrations. The left two adjunctions require injective model structures so that restriction preserves the objectwise weak equivalences and cofibrations. Then the overall zig-zag of Quillen equivalences involves a change of models equivalence between projective and injective model structures on $\widehat{RI}_{\mathcal{F}}^s\text{-mod}$.

Finally, we discuss cells. In all cases the cells are what we might call extended modules, which at each point are $R \wedge G/H_+$. However we need to bear in mind that over $\mathbf{LC}(G/K)$, geometric fixed points have been applied.

For the first adjunction, the cells come from the diagram $\mathbf{LI}^s(G)$, and the values on $\mathbf{LI}_F^s(G)$ are obtained by restriction: formally $(\text{res}_1 M)(x) = M(x)$ whenever x lies in $\mathbf{LI}_F^s(G)$. The comparison maps as F varies are thus the identity whenever they relate non-trivial entries. Accordingly, when we apply the right adjoint \lim_{\leftarrow} to $\text{res}_1 M$, at each point x we have a limit over a system eventually constant at $M(x)$, so the limit is the original value $M(x)$.

The second adjunction is rather trivial since the rings at the points $(G/K, n)_{G/K}$ are simply the limits over a finite diagram. Since the diagram is finite, these limits commute with smashing with G/H_+ , so the image of the cells of $RI_{\mathcal{F}}^s\text{-mod}$ are again the extended cells.

Finally, the real content is in the third adjunction. We need to argue that the image of the extended cells of $\widehat{RI}_{\mathcal{F}}^s\text{-mod}$ are the extended cells of $RI_*^s\text{-mod}$. At each point $(G/H, a)_{G/K}$ of $\mathbf{LI}^s(G)$ the relevant direct limits over F are eventually constant.

This leaves the points $(G/K, n)_{G/K}$ on the diagonal. It is to deal with these that the continuous limits were introduced, and we may apply the work of Section 8.

Proposition 10.6. *This process recovers the correct top left hand entry:*

$$\widetilde{R}_{top}(G/K, n)_{G/K} = \text{holim}'_{\leftarrow \mathbf{LCI}^*(G)_{G/K}} \widetilde{R}_{top}(G/H, a)_{G/K}.$$

Proof: First note that the diagram $\mathbf{LCI}(G)_{G/K}$ may be identified with the inflation of $\mathbf{LCI}(G/K)_{G/K}$, so that for notational simplicity we suppose $K = 1$.

What we must show is that the cofibres of the two left-hand horizontals in the diagram

$$\begin{array}{ccccc} S^0 & \longrightarrow & \text{holim}_{\leftarrow H \in \text{ConnSub}(\mathbf{G})^*} S^{\infty V(H)} & \longrightarrow & \Sigma E\mathcal{F}_+ \\ \downarrow & & \downarrow & & \downarrow \simeq \\ S^0 \wedge DE\mathcal{F}_+ & \longrightarrow & \text{holim}_{\leftarrow H \in \text{ConnSub}(\mathbf{G})^*} S^{\infty V(H)} \wedge DE\mathcal{F}_+ & \longrightarrow & \Sigma E\mathcal{F}_+ \wedge DE\mathcal{F}_+ \end{array}$$

are equivalent. The calculation of the cofibres is immediate from Lemma 8.8 since continuous inverse limits commute with smash products. The equivalence of the two cofibres follows since $S^0 \rightarrow DE\mathcal{F}_+$ is an \mathcal{F} -equivalence. \square

To summarize, we have the following.

Corollary 10.7. *There are Quillen adjunctions*

$$RI^s\text{-mod} \xrightleftharpoons[\lim_{\leftarrow}]{\text{res}_1} RI_{\mathcal{F}}^s\text{-mod} \xrightleftharpoons[\lim_{\leftarrow}^{\mathcal{F}}]{\text{res}_{\mathcal{F}}} \widehat{RI}_{\mathcal{F}}^s\text{-mod} \xrightleftharpoons[\text{res}_3]{\lim_{\rightarrow F}} RI_*^s\text{-mod}$$

These adjunctions induce Quillen equivalences on the associated cellularizations.

Part 4. Removing equivariance

11. FIXED POINT ADJUNCTIONS FOR MODULE CATEGORIES

We need to discuss passage to fixed points for modules, and it is convenient to collect together some generalities. Thus we suppose given a ring G -spectrum A , and we note that since Lewis-May fixed points are lax monoidal [36], A^K is a ring G/K -spectrum.

We need to pass between A -module G -spectra and A^K -module G/K -spectra, and we begin by explaining this in some detail.

11.A. Inflation and fixed points. First, we write $\pi : G \rightarrow G/K$ for the projection. Next, we suppose the G -spectra are indexed on the complete G -universe \mathcal{U} , and we use \mathcal{U}^K as our complete G/K -universe, writing $i : \mathcal{U}^K \rightarrow \mathcal{U}$ for the inclusion, so that we have an adjoint pair $i_* \vdash i^*$ of change of universe functors. The basic change of group functor is inflation,

$$\inf = \inf_{G/K}^G : G/K\text{-spectra}/\mathcal{U}^K \rightarrow G\text{-spectra}/\mathcal{U}$$

defined by

$$\inf Y = i_* \pi^* Y.$$

This is right adjoint to the Lewis-May fixed point functor

$$LM^K : G\text{-spectra}/\mathcal{U} \rightarrow G/K\text{-spectra}/\mathcal{U}^K$$

defined by

$$LM^K(X) = (i^* X)^K,$$

where $(\cdot)^K$ denotes termwise fixed points. Inflation and Lewis-May fixed points form a Quillen pair by [36, V.3.10].

It is usual to write X^K for the Lewis-May fixed points, leaving the context to imply the restriction to a K -fixed universe, and we shall generally follow this convention.

11.B. Inflation and fixed points for modules. Since passage to K -fixed points is lax monoidal, the K -fixed point spectrum of a ring G -spectrum A is a ring G/K -spectrum and the K -fixed point spectrum of an A -module is an A^K -module. Accordingly, we have a functor

$$\Psi^K : A\text{-mod-}G\text{-spectra} \rightarrow A^K\text{-mod-}G/K\text{-spectra},$$

where $\Psi^K M$ is M^K viewed as an A^K -module. The functor Ψ^K is also right adjoint. We will explain how to view the left adjoint as a composite of inflation and an extension of scalars. The purpose of the notation Ψ^K is to emphasize that we have not only taken K -fixed points, but also changed the ring over which we take modules. We continue to write X^K for the underlying K -fixed point functor.

First, we have explained that passage to Lewis-May K -fixed points is right adjoint to inflation. This lifts to an adjunction

$$\inf : A^K\text{-mod-}G/K\text{-spectra}/\mathcal{U}^K \rightleftarrows \inf A^K\text{-mod-}G\text{-spectra}/\mathcal{U} : LM^K.$$

To explain, since \inf is strong monoidal, $\inf A^K$ is a ring, and inflation takes A^K -modules to $\inf A^K$ -modules. On the other hand, since $(\cdot)^K$ is lax monoidal, it takes $\inf A^K$ -modules to $(\inf A^K)^K$ -modules, which are then regarded as A^K -modules, by restriction along the unit map

$$\eta : A^K \rightarrow (\inf A^K)^K.$$

Since the Lewis-May fixed points functor is a right Quillen functor on the underlying categories of orthogonal spectra by [36, V.3.10], this is a Quillen adjunction on the module categories by [36, III.7.6 (i)].

Second, inclusion of subspectra gives a ring map

$$\pi^* A^K \longrightarrow i^* A,$$

which corresponds to the counit

$$\theta : \inf A^K \longrightarrow A.$$

We therefore have by [36, III.7.6 (vi)] an associated Quillen adjunction for restriction and extension of scalars:

$$\theta_* : \inf A^K\text{-mod-}G\text{-spectra}/\mathcal{U} \rightleftarrows A\text{-mod-}G\text{-spectra}/\mathcal{U} : \theta^* .$$

We summarize the discussion.

Proposition 11.1. *There is a Quillen adjunction*

$$A \otimes_{\inf A^K} \inf(\cdot) : A^K\text{-mod-}G/K\text{-spectra} \rightleftarrows A\text{-mod-}G\text{-spectra} : \Psi^K ,$$

where $\Psi^K(M) = M^K$ viewed as an A^K -module. □

There are a number of interesting examples, and in some of those directly relevant to us the adjunction is a Quillen equivalence.

Example 11.2. Perhaps the first example has $A = \mathbb{S}$ and $K = G$. We first observe that by Segal-tom Dieck splitting we have

$$A^G \simeq \bigvee_{(H)} BW_G(H)^{L(H)},$$

where $L(H)$ is the representation on the tangent space to the identity coset of G/H . This is rather a complicated spectrum, and quite different from the sphere.

When G is finite, and we work rationally, A^G is equivalent to a product of spheres indexed by the conjugacy classes of subgroups, in fact to the Eilenberg-MacLane spectrum for the rational Burnside ring. On the left, the rational Burnside ring has another role. Using its idempotents, the equivariant sphere S^0 also splits into pieces $e_H S^0$ also corresponding to conjugacy classes of subgroups. However the Quillen adjunction is not a Quillen equivalence. Indeed, we see that the factors corresponding to H are usually inequivalent: the category of G -equivariant $e_H S^0$ -modules is equivalent to the category of $\mathbb{Q}W_G(H)$ -modules, and the category of non-equivariant S^0 -modules is equivalent to the category of \mathbb{Q} -modules. If $W_G(H)$ is non-trivial, there is more than one simple $\mathbb{Q}W_G(H)$ -module. The Cellularization Principle gives an equivalence between the trivial modules and $A(G)$ -modules.

When G is the circle group one does not get a Quillen equivalence either (by connectivity the counit is not an equivalence for a free cell G_+).

Example 11.3. The first example of direct interest to us is $A = S^{\infty V(K)}$ with $A^K = S^0$. We remind the reader that in this case A is not a ring, and that the category of A -modules means the category of G -spectra over K . We shall see that in this case the Quillen adjunction is a Quillen equivalence, as is familiar to devotees of geometric fixed points.

Example 11.4. An example of rather a different character is $A = DEG_+$ with $K = G$ and $A^G = DBG_+$. We shall see that in this case the Quillen adjunction is a Quillen equivalence. This might be called the Eilenberg-Moore equivalence, and may be viewed as the prototype for the strategy of the present paper.

Example 11.5. Generalizing the previous example, we may take $A = DEF_+$. This example is also discussed at length in the sequel.

Example 11.6. Combining two of the previous examples, we reach $A = DEF_+ \wedge S^{\infty V(K)}$ with $A^K = \Phi^K DEF_+$. This example is discussed at length in the sequel.

11.C. Towards Quillen equivalences. We would like to compare the category of module G -spectra over A to the category of module G/K -spectra over A^K . The Quillen adjunction will generally not be a Quillen equivalence, but by the Cellularization Principle we will obtain some useful results.

In somewhat simplified notation, consider the unit and counit of the derived adjunction

$$\eta : Y \longrightarrow (A \otimes_{\inf A^K} \inf Y)^K \text{ and } \epsilon : A \otimes_{\inf A^K} (\inf X^K) \longrightarrow X.$$

We see that η is an equivalence for $Y = A^K$ and that ϵ is an equivalence for $X = A$, so that we always have an equivalence

$$A\text{-cell-}A\text{-mod-}G\text{-spectra} \simeq A^K\text{-cell-}A^K\text{-mod-}G/K\text{-spectra}.$$

However, A is not usually a generator of the category of equivariant A -modules since we also need the modules of the form $A \wedge G/H_+$ for proper subgroups H , and similarly for A^K -modules.

The equivalences between cellularized categories are often interesting, but there are important cases where sets of generators correspond under the adjunction. The Cellularization Principle gives an equivalence between cellularizations with respect to these sets of generators. Since cellularization with respect to a set of generators has no effect, in this case we obtain an equivalence between A -module G -spectra and A^K -module G/K -spectra.

In the first case (see Subsection 11.D), we will use all the generators $A^K \wedge \overline{G}/\overline{H}_+$, and restrict A so that their images give a set of generators of the category of A -modules. In the second case (see Subsection 13.B), we take $K = G$ so that A^G is the generator of the category of A^G -modules. There are then a number of special cases (for example when A is complex orientable and G is a torus) where A does generate all A -modules.

11.D. Geometric fixed points for ring and module spectra. A very satisfying class of examples is a generalization of the basic property of geometric fixed point spectra. The contents of this section applies to any compact Lie group G and normal subgroup K , though in general $S^{\infty V(K)}$ must be replaced by $\tilde{E}[\not\supseteq K]$. The relationship between Lewis-May fixed points and geometric points is described in [33, III.9].

Recall that the *geometric isotropy* of a G -spectrum X is the set

$$\mathcal{GI}(X) := \{H \mid \pi_*(\Phi^H X) \neq 0\}$$

of subgroups contributing to the homotopy theory of X . We say that X *lies over* K if every subgroup in $\mathcal{GI}(X)$ contains K . It follows from the geometric fixed point Whitehead Theorem that X lies over K if and only if

$$X \simeq S^{\infty V(K)} \wedge X$$

that for any X

$$S^{\infty V(K)} \wedge X \simeq S^{\infty V(K)} \wedge \inf \Phi^K X,$$

and that if Y lies over K then

$$[X, Y]^G \cong [\Phi^K X, \Phi^K Y]^{G/K}.$$

All of this generalizes easily to categories of modules.

Theorem 11.7. *If A is a ring G -spectrum concentrated over K then the Quillen adjunction of Proposition 11.1 is a Quillen equivalence*

$$A\text{-mod-}G\text{-spectra} \simeq A^K\text{-mod-}G/K\text{-spectra}.$$

Remark 11.8. (i) If X is an A -module, then X is a retract of $A \wedge X$, so that if A lies over K so too do its modules.

(ii) If X lies over K then $X^K \simeq \Phi^K X$. Accordingly $A^K \simeq \Phi^K A$ and Lewis-May fixed points may be replaced by geometric fixed points throughout.

(iii) It is appropriate to recall the well known equivalence

$$G\text{-spectra}/K \simeq G/K\text{-spectra}.$$

This can be included in the statement of the theorem if we permit $A = S^{\infty V(K)}$. Since A is not a ring spectrum we interpret A -module G -spectra as G -spectra over K in the usual way, and the category of module G/K -spectra over $A^K = S^0$ is just the category of G/K -spectra.

Proof: As explained in Subsection 11.C above, we need only check that the unit and counit are equivalences, and this only needs to be done for generators of the categories. Before we do this, observe that since A is concentrated over K ,

$$A \wedge X \simeq S^{\infty V(K)} \wedge \inf A^K \wedge \inf \Phi^K X.$$

For the unit, we need only check equivalence for $Y = A^K \wedge \overline{G}/\overline{H}_+$, where \overline{H} runs through the closed subgroups of $\overline{G} = G/K$. But in this case we have

$$\eta : A^K \wedge \overline{G}/\overline{H}_+ \longrightarrow [A \otimes_{\inf A^K} \inf (A^K \wedge \overline{G}/\overline{H}_+)]^K.$$

This map is obtained from the unit for $A_0 = S^{\infty V(K)}$ by extending scalars from $A_0^K = S^0$ to A^K on both sides, and is therefore an equivalence.

For the counit, we need only check equivalence for $X = A \wedge G/H_+$, where H runs through the closed subgroups of G . But in this case we have

$$\epsilon : A \otimes_{\inf A^K} \inf [(A \wedge G/H_+)^K] \longrightarrow A \wedge G/H_+$$

This is obtained from the counit for A_0 by extending scalars from $A_0 = S^{\infty V(K)}$ to $A = S^{\infty V(K)} \wedge A$ on both sides, and is therefore an equivalence. \square

12. VERTICES

Section 9 shows that a G -spectrum can be built from a diagram whose vertices are module categories, and it is natural to ask what each module category contributes to the G -spectrum. In this section we shall see that on their own, the vertices each give enormous categories with little geometric significance. Only when combined with the rest of the diagram do they come under control.

12.A. Change of groups. To begin with, we have a completely satisfactory description of the piece at $(G/K, n)$. Indeed, this is the special case $A = \tilde{R}_{top}(G/K, n) = S^{\infty V(K)}$, and $A^K = S^0$, which is dealt with in Remark 11.8 (iii); in the present language, it states that there is a Quillen equivalence

$$S^{\infty V(K)}\text{-mod-}G\text{-spectra} \simeq G/K\text{-spectra}.$$

For the vertex $(G/K, c)$, this step only goes part of the way. Applying Theorem 11.7 with

$$A = \tilde{R}_{top}(G/K, c) = S^{\infty V(K)} \wedge DEF_+,$$

we obtain a reduction to G/K -spectra.

Corollary 12.1. *There is a Quillen equivalence*

$$S^{\infty V(K)} \wedge DEF_+\text{-mod-}G\text{-spectra} \simeq \Phi^K DEF_+\text{-mod-}G/K\text{-spectra}. \quad \square$$

In order to use induction on the dimension of the group, we really want to express information at the $(G/K, c)$ vertex in terms of DEF/K_+ -module G/K -spectra. Now there is a comparison map $DEF/K_+ \rightarrow \Phi^K DEF_+$ of ring G/K -spectra, but the codomain is enormously bigger in the sense that on coefficients the map is $\mathcal{O}_{\mathcal{F}/K} \rightarrow \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$. This is quite well behaved, in terms of flatness, but it is very far from being an equivalence. In effect we avoid the necessity of considering this, since the presence of the map $\tilde{R}_{top}(G/K, n) \rightarrow \tilde{R}_{top}(G/K, c)$ lets us recognize modules induced from $DEF/K_+\text{-mod-}G/K$ -spectra.

12.B. Almost free spectra. Arbitrary modules over the ring at $(G/K, c)$ are not closely related to equivariant homotopy theory, but those which become trivial at $(G/H, c)$ when H is strictly bigger than K (which we might call *torsion modules*) do have a geometric interpretation that we describe here.

A G -space is said to be *almost free* if all isotropy groups are finite. Up to weak equivalence, these are the spaces so that the universal map induces an equivalence $X \simeq EF_+ \wedge X$, and we use this to define geometrically almost free spectra. A suitable model may be obtained by taking all spectra and taking the weak equivalences to be maps which are F -equivalences for all finite subgroups F ; see [36, IV.6.5].

Proposition 12.2. *There are Quillen equivalences*

$$\mathcal{F}/K\text{-cell-}DEF/K_+\text{-mod-}G/K\text{-spectra} \simeq \mathcal{F}/K\text{-cell-}G/K\text{-spectra}$$

where cellularization on both sides is with respect to the images of the cells $\overline{G}/\overline{H}$ with finite isotropy (i.e., with $\overline{H} \in \mathcal{F}/K$); the category on the right is that of almost free G/K -spectra.

Proof: There is a ring map

$$\eta : \mathbb{S} \rightarrow DEF/K_+$$

This give an induction-restriction adjunction

$$\eta^* : DEF/K_+\text{-mod-}G/K\text{-spectra} \rightleftarrows G/K\text{-spectra} : \eta_*$$

between categories of module G/K -spectra. The category of DEF/K_+ -modules is enormous, having many objects which are not extended along η from G/K -spectra, and the category of $\Phi^K DEF_+$ -modules is even bigger.

On the other hand the relevant unit and counit of the adjunction are

$$P \longrightarrow DEF/K_+ \wedge P \text{ and } DEF/K_+ \wedge N \longrightarrow N,$$

where P is an \mathbb{S} -module and N is an DEF/K_+ -module. Since EF/K_+ is \mathcal{F}/K -equivalent to \mathbb{S} , it is clear that the unit is an equivalence when P is built from cells G/\tilde{K}_+ in \mathcal{F}/K , and that the counit is an equivalence when N is built from cells $DEF/K_+ \wedge (G/K)/(\tilde{K}/K)_+$. \square

13. FIXED POINT EQUIVALENCES FOR MODULE CATEGORIES

The next step is to replace the $\mathbf{LI}(G)$ -diagram \tilde{R}_{top} of rings in the category of G -spectra by an $\mathbf{LI}(G)$ -diagram R_{top} of non-equivariant ring spectra by passage to fixed points. Similarly, for module categories we replace the category $\tilde{R}_{top}\text{-mod-}G\text{-spectra}$ of diagrams of \tilde{R}_{top} -modules in G -spectra by diagrams of R_{top} -modules in the category of non-equivariant spectra.

Our constructions operate separately on each point in the diagram, and maps of diagrams that are weak equivalences of each point in the diagram are weak equivalences of diagrams. This means that all verifications can be made in categories of module spectra over individual ring spectra.

13.A. The diagram R_{top} . In fact we define R_{top} by taking fixed points on the $\mathbf{LI}(G)$ -diagram \tilde{R}_{top} . Without changing notation, we first replace \tilde{R}_{top} by its fibrant replacement in the diagram-injective model category of $\mathbf{LI}(G)$ -diagrams of commutative ring G -spectra [27, 5.1.3]. By Lemma B.5 the category of modules over this fibrant replacement is Quillen equivalent to the original category $\tilde{R}_{top}\text{-mod-}G\text{-spectra}$. We then apply fixed points termwise, so that

$$R_{top}(G/K, c)_{G/L} := (\tilde{R}_{top}(G/K, c))^{G/L} = (S^{\infty V(K/L)} \wedge DEF/L_+)^{G/L}.$$

These are all non-equivariant commutative rational ring spectra, and therefore commutative $H\mathbb{Q}$ -algebras. The category of commutative $H\mathbb{Q}$ -algebras is equivalent to the category of commutative DGAs over \mathbb{Q} and the module categories over corresponding algebras are equivalent [45]. To aid comparisons, we will name the resulting ring spectra by their homotopy, so that for a graded ring A we write A_{top} for a rational ring spectrum with $\pi_*(A_{top}) = A$. We will show in Section 15 that the ring spectra that occur in R_{top} are all intrinsically formal, so that the notation should not be misleading. For the value at $(G/K, c)$ we have

$$R_{top}(G/K, c) = (DEF_+ \wedge S^{\infty V(K)})^G = \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}}^{top}.$$

We will use a special property of the ring R_{top} .

Definition 13.1. The category $\mathbf{LI}^{split}(G)$ is obtained from the category $\mathbf{LI}(G)$ by adjoining a transitive system of maps

$$\sigma_{G/K}^{G/L} : (G/K, c)_{G/K} \longrightarrow (G/L, c)_{G/L}$$

along the leading diagonal whenever $G \supseteq K \supseteq L$.

An $\mathbf{LI}(G)$ -diagram X in \mathbb{C} is *split* if it extends to an $\mathbf{LI}^{split}(G)$ -diagram X^{split} :

$$\begin{array}{ccc} \mathbf{LI}(G) & \xrightarrow{X} & \mathbb{C}. \\ \downarrow & \nearrow X^{split} & \\ \mathbf{LI}^{split}(G) & & \end{array}$$

Now observe that the diagram R_{top} is split. In fact the maps

$$R_{top}(G/K, c)_{G/K} \longrightarrow R_{top}(G/L, c)_{G/L}$$

come from inflation maps we have seen before; indeed the map

$$(DEF/K_+)^{G/K} \longrightarrow (DEF/L_+)^{G/L}.$$

is obtained by passing to G/K -fixed points from

$$DEF/K_+ \longrightarrow (DEF/L_+)^{K/L}$$

which corresponds to the G/L -equivariant map

$$\inf_{G/K}^{G/L} DEF/K_+ \longrightarrow DEF/L_+$$

whose adjoint

$$EF/L_+ \wedge \inf_{G/K}^{G/L} DEF/K_+ \longrightarrow S^0$$

is obtained by composing the G/L -map $EF/L_+ \longrightarrow EF/K_+$ with evaluation.

13.B. The Quillen equivalence. Assembling our progress, we have a rather remarkable equivalence between diagrams of module G -spectra and diagrams of non-equivariant module spectra.

Theorem 13.2. *There is a Quillen equivalence*

$$\theta_* : \text{cell-}R_{top}\text{-mod-spectra} \xrightleftharpoons{\quad} \text{cell-}\tilde{R}_{top}\text{-mod-}G\text{-spectra} : \Psi^G ,$$

where the modules on the left are in the category of spectra indexed on \mathcal{U}^G and those on the right are in the category of G -spectra indexed in \mathcal{U} . In both cases, cellularization is with respect to the images of the G -spaces G/H_+ as H runs through closed subgroups.

Remark 13.3. To see the content of this, we note that there is a counterpart for free spectra. In this case $\tilde{R}_{top} = DEG_+$, and $R_{top} = DBG_+$. The corresponding statement is that there is a Quillen equivalence

$$\theta_* : \text{cell-}DBG_+\text{-mod-spectra} \xrightleftharpoons{\quad} \text{cell-}DEG_+\text{-mod-}G\text{-spectra} : \Psi^G ,$$

where the modules on the left are in the category of spectra indexed on \mathcal{U}^G and those on the right are in the category of G -spectra indexed in \mathcal{U} . The cells in question are the images of free cells, which is to say G_+ on the right and $(G_+)^G = S^d$ on the left. In particular, because G_+ is free, the universe \mathcal{U} could be replaced by the trivial universe \mathcal{U}^G .

We shall see that when G is a torus, the cellularization may be omitted on both sides in this example, but this is a special feature of the torus (for example, when G is a non-trivial finite group some cellularization is essential).

Proof: This is another example of the general setup of Section 11. Throughout the proof we work in the derived category and omit notation for inflation for simplicity, since the context shows where it is required. For convenience we record the unit and counit of the adjunction in the present context:

13.4.

$$\eta : N \longrightarrow (\tilde{R}_{top} \otimes_{R_{top}} N)^G \text{ and } \epsilon : \tilde{R}_{top} \otimes_{R_{top}} (M^G) \longrightarrow M.$$

We must show that these are equivalences for all cellular M and N . More explicitly, both the unit and counit are maps of $\mathbf{LI}(G)$ -diagrams. We must show that they are equivalences at the vertices $(G/L, c)_{G/K}$ whenever $L \subseteq K$.

13.B.1. *The unit is an equivalence.* To start with, it is straightforward to see that η is an equivalence for all cellular N . Indeed, there is a similar Quillen adjunction before cellularization. We will check the unit of this adjunction is an equivalence for all N and then cellularize to give the desired statement.

We may verify the unit is an equivalence by checking at each point of the diagram. In effect we are verifying that

$$\eta' : N' \longrightarrow (\tilde{R}' \otimes_{R'} N')^G$$

is an equivalence for all R' -modules N' , where $R' = (\tilde{R}')^G$. The category of (non-equivariant) R' -module spectra is generated by R' itself, so it suffices to observe that η' is an isomorphism for $N' = R'$ and that the class of spectra N' for which it is an equivalence is closed under cofibre sequences and coproducts. Since this holds for each point $(G/K, a)$ in the diagram (i.e., for $\tilde{R}' = \tilde{R}_{top}(G/K, a)$) it follows that the unit η is itself an equivalence, and by cellularization that the original unit map in 13.4 is an equivalence as required.

13.B.2. *The structure of the proof for the counit.* It remains to show that the counit is an equivalence at $(G/K, c)_{G/L}$ whenever $L \subseteq K$. We argue by induction on the dimension of G . Since the statement is trivial for the trivial group, this induction starts.

In terms of the $\mathbf{LI}(G)$ -diagram, our inductive hypothesis allows us to assume that the counit is an equivalence at $(G/K, c)_{G/L}$ whenever $K \neq 1$, and we need to prove the result at $(G/K, c) = (G/K, c)_{G/1}$ for all K .

The rest of the proof has two parts: the fact that the counit is an equivalence at $(G/1, c)$ and the argument by induction and change of groups that equivalence at $(G/K, c)$ for $K \neq 1$ follows from the results in previous rows. Both steps are quite substantial.

13.B.3. *The counit 13.4 is an equivalence at $(G/1, c)$.* We shall in fact show that in this case the equivalence holds without cellularization, because the cells used on the two sides are generators.

It is convenient to replace the cells G/H_+ by dual cells $D(G/H_+)$. Since G is abelian, $\Sigma^{\dim(G/H)} D(G/H_+) \simeq G/H_+$, so this is harmless. In fact we will argue directly that the counit is an equivalence when M is the image of DX for any finite G -spectrum X .

We begin by making explicit what must be proved. To start with,

$$\tilde{R}_{top}(G/1, c) = DE\mathcal{F}_+ \simeq \prod_{F \in \mathcal{F}} DE\langle F \rangle \text{ and } R_{top}(G/1, c) = D(E\mathcal{F}_+)^G \simeq \prod_{F \in \mathcal{F}} D(BG/F_+).$$

Next note that the \tilde{R}_{top} -module M corresponding to the G -spectrum DX has the value $M(G/1, c) = DE\mathcal{F}_+ \wedge DX$ at $(G/1, c)$, so that in the domain we have

$$DE\mathcal{F}_+ \otimes_{(DE\mathcal{F}_+)^G} [D(E\mathcal{F}_+) \wedge DX]^G \simeq DE\mathcal{F}_+ \otimes_{(DE\mathcal{F}_+)^G} [D(E\mathcal{F}_+ \wedge X)]^G.$$

Now consider the class of G -spectra X for which the counit at $(G/1, c)$

$$\epsilon(G/1, c) : DE\mathcal{F}_+ \otimes_{(DE\mathcal{F}_+)^G} [D(E\mathcal{F}_+ \wedge X)]^G \longrightarrow D(E\mathcal{F}_+ \wedge X)$$

is an equivalence.

Lemma 13.5. *The class of X for which the counit $\epsilon(G/1, c)$ is an equivalence is closed under cofibre sequences and suspensions by representations.*

Proof: Closure under cofibre sequences is obvious.

Closure under suspension is based on the Thom isomorphism, which in turn comes from the splitting $E\mathcal{F}_+ \simeq \bigvee_F E\langle F \rangle$ [12] which immediately gives a splitting

$$D(E\mathcal{F}_+ \wedge X) \simeq \prod_F D(E\langle F \rangle \wedge X).$$

Accordingly, $\epsilon(G/1, c)$ is an equivalence providing we have the corresponding splitting for its domain. This amounts to showing that the natural comparison map

$$\left[\prod_F DE\langle F \rangle \right] \otimes_{\prod_F (DE\langle F \rangle)^G} \left[\prod_F (D(E\langle F \rangle \wedge X))^G \right] \longrightarrow \left[\prod_F DE\langle F \rangle \otimes_{(DE\langle F \rangle)^G} (D(E\langle F \rangle \wedge X))^G \right]$$

is an equivalence.

Now suppose given a representation V and use the Thom isomorphism

$$S^V \wedge E\langle F \rangle \simeq S^{|V^F|} \wedge E\langle F \rangle$$

for each finite subgroup F . We divide the finite subgroups into sets according to $\dim_{\mathbb{C}}(V^F)$. Of course there are only finitely many of these, and we may apply the corresponding orthogonal idempotents to consider the sets separately. For the class of F with $\dim_{\mathbb{C}}(V^F) = k$ we have

$$\prod_F (D(E\langle F \rangle \wedge S^V \wedge X))^G \simeq \Sigma^{-k} \prod_F (D(E\langle F \rangle \wedge X))^G,$$

so both sides of the equation are simply desuspended by k . □

Now observe that $\epsilon(G/1, c)$ is obviously an equivalence for $X = S^0$, and it remains to show that Lemma 13.5 implies that $\epsilon(G/1, c)$ is an equivalence for all finite spectra.

Lemma 13.6. *When G is a torus, the G -spectrum S^0 builds all finite G -spectra using cofibre sequences and suspensions by representations.*

Proof: It suffices to construct all spectra G/H_+ .

Consider a non-trivial 1-dimensional representation α , with kernel K . We have $S(\alpha) = G/K$, and therefore a cofibre sequence

$$G/K_+ \longrightarrow S^0 \longrightarrow S^\alpha,$$

showing that the map is an isomorphism for spectra of the form G/K_+ . More generally if S^0 builds X , we see it also builds $X \wedge G/K_+$.

The subgroups occurring as kernels K as in the previous paragraph are precisely those of the form $K = H \times C$ where H is a rank $r - 1$ torus and C is a finite cyclic group. Finally we note that an arbitrary subgroup H we have

$$G/H = G/K_1 \times G/K_2 \times \cdots \times G/K_r$$

for suitable K_i occurring as kernels. \square

Remark 13.7. To see the content of the equivalence of $\epsilon(G/1, c)$, apply idempotents to take the factor with $F = 1$. Note that for any X we have $[D(EG_+ \wedge X)]^G \simeq D(EG_+ \wedge_G X)$ and we need to verify

$$DEG_+ \otimes_{DBG_+} D(EG_+ \wedge_G X) \xrightarrow{\simeq} D(EG_+ \wedge X)$$

for the based finite complex X . This is just the based counterpart of the Eilenberg-Moore spectral sequence for the fibration

$$Y \longrightarrow EG \times_G Y \longrightarrow EG \times_G * = BG,$$

for the unbased space Y , which converges since G is connected.

13.B.4. *The counit 13.4 is an equivalence at $(G/K, c)$ for $K \neq 1$.* We will deduce that the counit is an equivalence at $(G/K, c)$ from the fact that the counit for the quotient group G/K is an equivalence at $((G/K)/(K/K)1, c)_{G/K}$.

Taking G -fixed points can be broken into two steps: taking K -fixed points followed by taking G/K -fixed points. The adjunction between Ψ^G and inflation is similarly a composite. Theorem 11.7 with $A = \tilde{R}_{top}(G/K, c)$ shows that $\tilde{R}_{top}(G/K, c)$ -module G -spectra are equivalent to $\Phi^K DE\mathcal{F}_+$ -module spectra, and in particular that the counit of the K -fixed point adjunction is an equivalence. This gives the reduction to a statement about modules over $\Phi^K DE\mathcal{F}_+$.

Lemma 13.8. *Provided*

$$\Phi^K DE\mathcal{F}_+ \otimes_{(\Phi^K DE\mathcal{F}_+)^{G/K}} P^{G/K} \simeq P$$

for all cellular $\Phi^K DE\mathcal{F}_+$ -module G/K -spectra P , the counit 13.4 is an equivalence at $(G/K, c)$ for all cellular $\tilde{R}_{top}(G/K, c)$ -module spectra M .

Proof: First note that

$$(S^{\infty V(K)} \wedge DE\mathcal{F}_+)^G \simeq ((S^{\infty V(K)} \wedge DE\mathcal{F}_+)^K)^{G/K} \simeq (\Phi^K DE\mathcal{F}_+)^{G/K}.$$

The main input comes from Theorem 11.7, which states

$$(S^{\infty V(K)} \wedge DE\mathcal{F}_+) \otimes_{\Phi^K DE\mathcal{F}_+} M^K \simeq M.$$

We therefore only need to show that the left-hand side is equivalent to the domain of the counit 13.4, namely

$$\begin{aligned} (S^{\infty V(K)} \wedge DE\mathcal{F}_+) \otimes_{(S^{\infty V(K)} \wedge DE\mathcal{F}_+)^G} M^G &\simeq \\ (S^{\infty V(K)} \wedge DE\mathcal{F}_+) \otimes_{\Phi^K DE\mathcal{F}_+} \Phi^K DE\mathcal{F}_+ \otimes_{(\Phi^K DE\mathcal{F}_+)^{G/K}} (M^K)^{G/K} \end{aligned}$$

The required equivalence follows from the hypothesis by taking $P = M^K$, and inducing along $\Phi^K DE\mathcal{F}_+ \longrightarrow S^{\infty V(K)} \wedge DE\mathcal{F}_+$.

We should also remark that notions of cellularity correspond. Indeed, to check $\epsilon(G/K, c)$ is an equivalence for cellular M we need only consider $M = DEF_+ \wedge S^{\infty V(K)} \wedge DX$ for finite G -spectra X . This has K -fixed points $M^K = \Phi^K DEF_+ \wedge D(\Phi^K X)$, which is cellular. \square

It remains to reduce to a statement about modules over the ring G/K -spectrum DEF/K_+ . The starting point of the reduction is that the module spectra P that occur are all of the form

$$P = \Phi^K DEF_+ \otimes_{DEF/K_+} Q,$$

induced from DEF/K_+ -module G/K -spectra. The second ingredient is the very special nature of the map $DEF/K_+ \rightarrow \Phi^K DEF_+$ of ring G/K -spectra.

Lemma 13.9. *Provided*

$$DEF/K_+ \otimes_{(DEF/K_+)^{G/K}} Q^{G/K} \simeq Q$$

for all cellular DEF/K_+ -module spectra Q , the hypothesis of Lemma 13.8 is satisfied and hence the counit 13.4 is an equivalence at $(G/K, c)$ for all cellular $\tilde{R}_{top}(G/K, c)$ -module spectra M .

Proof: It will aid readability to simplify notation. Thus we write

$$\begin{array}{ccc} A^{\overline{G}} & \longrightarrow & A \\ \uparrow & & \uparrow \text{ for } \\ B^{\overline{G}} & \longrightarrow & B \end{array} \quad \begin{array}{ccc} (\Phi^K DEF_+)^{G/K} & \longrightarrow & \Phi^K DEF_+ \\ \uparrow & & \uparrow \\ (DEF/K_+)^{G/K} & \longrightarrow & DEF/K_+ \end{array}$$

Thus we have $P = A \otimes_B Q$, and the hypothesis of the lemma is an equivalence $B \otimes_{B^{\overline{G}}} Q^{\overline{G}} \simeq Q$. It is clear that if $Q = B \wedge DY$ then $P = A \wedge DY$, so that notions of cellularity correspond.

We will establish equivalences

$$A \otimes_{A^{\overline{G}}} P^{\overline{G}} = A \otimes_{A^{\overline{G}}} [A \otimes_B Q]^{\overline{G}} \stackrel{(a)}{\simeq} A \otimes_{A^{\overline{G}}} A^{\overline{G}} \otimes_{B^{\overline{G}}} Q^{\overline{G}} \stackrel{(b)}{\simeq} A \otimes_B B \otimes_{B^{\overline{G}}} Q^{\overline{G}} \stackrel{(c)}{\simeq} A \otimes_B Q = P.$$

This will complete the proof since the composite equivalence establishes the hypothesis of Lemma 13.8.

The equivalence (c) is obtained from the equivalence stated as the hypothesis of the lemma by inducing along $B \rightarrow A$, and the equivalence (b) uses formal properties of the tensor product.

For the equivalence (a) it suffices to establish that the natural map

$$[A \otimes_B Q]^{\overline{G}} \xleftarrow{\simeq} A^{\overline{G}} \otimes_{B^{\overline{G}}} Q^{\overline{G}}.$$

is an equivalence, since we may then apply $A \otimes_{A^{\overline{G}}} (\cdot)$. For this we note that the result is clear for $Q = B$ and use an argument similar to the one in Subsubsection 13.B.3. Indeed, by an argument precisely similar to that of Lemma 13.5 we have an equivalence for $Q = B \wedge S^W$ (we split $B = DEF/K_+$ into a finite product on each of which smashing with S^W gives a non-equivariant suspension). It then follows from Lemma 13.6, we have an equivalence for $Q = B \wedge G/H_+$ for any $H \supseteq K$ as required. \square

By induction on the dimension of the group, the counit at $(G/1, c)$ is an equivalence for the quotient group G/K . This is the hypothesis of Lemma 13.9, and its conclusion gives

the hypothesis of Lemma 13.8. Accordingly, Lemma 13.8 shows that the counit 13.4 is an equivalence at $(G/K, c)$ for G .

The counit 13.4 is therefore an equivalence at all points of the diagram, and hence it is an equivalence. This completes the proof of Theorem 13.2. \square

Part 5. Algebraic equivalences

14. FROM SPECTRA TO DGAs

In this section we use the results from [46] to show that the category of module spectra over the diagram R_{top} of commutative ring spectra is Quillen equivalent to a category of differential graded modules over a diagram R_t of commutative DGAs. It then follows that the cellularizations of these model categories are also Quillen equivalent. Since [46] works with symmetric spectra, we apply the functor \mathbb{U} from [37] to the split $\mathbf{LI}(G)$ -diagram R_{top} of orthogonal ring spectra to obtain a split $\mathbf{LI}(G)$ -diagram $\mathbb{U}R_{top}$ of symmetric ring spectra. Now by [37, 0.6] there is a Quillen equivalence between $R_{top}\text{-mod}$ in orthogonal spectra and $\mathbb{U}R_{top}\text{-mod}$ in symmetric spectra.

In more detail, in [46, 1.1] a composite functor Θ is defined which produces a Quillen equivalence between $H\mathbb{Z}$ -algebra spectra (with many objects) and DGAs (with many objects). Given an $H\mathbb{Z}$ -algebra spectrum (with many objects), B , it is shown in [46, 2.15] that the category of module spectra over B is Quillen equivalent to the category of differential graded modules over ΘB , a DGA (with many objects). Furthermore, there is a second functor Θ' which is symmetric monoidal, so that it takes commutative rings spectra to commutative DGAs. Finally, over the rationals the two functors are naturally equivalent, so that by [46, 1.2], if B is a commutative $H\mathbb{Q}$ -algebra then ΘB is naturally weakly equivalent to a commutative DGA $\Theta' B$.

Definition 14.1. Applying functors to the split $\mathbf{LI}(G)$ -diagram of commutative rational ring spectra R_{top} , we define R_t to be the split $\mathbf{LI}(G)$ -diagram $\Theta'(H\mathbb{Q} \wedge \mathbb{U}R_{top})$ of commutative DGAs.

As in Remark B.7, we make no distinction between this diagram and the associated enriched category over spectra or over chain complexes. Note, throughout this section we are implicitly considering the standard (or diagram projective) model structures from [43, A.1.1] on modules over rings with many objects.

Proposition 14.2. *There is a zig-zag of Quillen equivalences between the category of module spectra $R_{top}\text{-mod}$ and the category of differential graded modules $R_t\text{-mod}$.*

Proof: As mentioned above the first step is a Quillen equivalence between $R_{top}\text{-mod}$ over orthogonal spectra and $\mathbb{U}R_{top}\text{-mod}$ over symmetric spectra by [37, 0.6]. Since R_{top} is rational, the unit map $\mathbb{U}R_{top} \rightarrow H\mathbb{Q} \wedge \mathbb{U}R_{top}$ is a weak equivalence which induces a Quillen equivalence on the associated module categories by extension and restriction of scalars. Combining these steps with [46, 2.15] produces a Quillen equivalence between $R_{top}\text{-mod}$ and $\Theta(H\mathbb{Q} \wedge \mathbb{U}R_{top})\text{-mod}$. Since $H\mathbb{Q} \wedge \mathbb{U}R_{top}$ is a diagram of commutative $H\mathbb{Q}$ -algebras, it follows that $\Theta'(H\mathbb{Q} \wedge \mathbb{U}R_{top})$ is a diagram of commutative rational DGAs which is weakly equivalent to the diagram $\Theta(H\mathbb{Q} \wedge \mathbb{U}R_{top})$. By [43, A.1.1], extension and restriction of scalars over these

weak equivalences produce the last steps in the stated zig-zag of Quillen equivalences. \square

Corollary A.7 below shows that cellularization preserves zig-zags of Quillen equivalences as long as the cells in the target category are taken to be the images under the relevant derived functors of the cells in the source category. Here we begin with the cellularization of $R_{top}\text{-mod}$ with respect to the images of G/H_+ as H runs through closed subgroups. Then, at each of the next steps, the cells are the images of G/H_+ under the appropriate derived functor.

Corollary 14.3. *There is a zig-zag of Quillen equivalences between the cellularizations of the model categories in Proposition 14.2; that is, $\text{cell-}R_{top}\text{-mod-spectra}$ and $\text{cell-}R_t\text{-mod-spectra}$ are Quillen equivalent.*

15. RIGIDITY

The outcome of the process described in Section 14 is the split $\mathbf{LI}(G)$ -diagram R_t (Definition 14.1) of commutative DGAs, and we may form the corresponding category $R_t\text{-mod}$ of DG modules over R_t . More explicitly, R_t is a split $\mathbf{LI}(G)$ -diagram, and the $G/1$ -row is of the form

$$R_t(G/1, c) \longrightarrow R_t(G/H_1, c) \longrightarrow R_t(G/H_2, c) \longrightarrow \cdots \longrightarrow R_t(G/G, c)$$

and its homology $R_a = H^*(R_t)$ has as its $G/1$ -row the diagram

$$\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}_{H_1}^{-1} \mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}_{H_2}^{-1} \mathcal{O}_{\mathcal{F}} \longrightarrow \cdots \longrightarrow \mathcal{E}_G^{-1} \mathcal{O}_{\mathcal{F}}$$

Proposition 15.1. *The diagram R_a is intrinsically formal in the sense that any split $\mathbf{LI}(G)$ -diagram R_t of commutative DGAs with $H_*(R_t) \cong R_a$ is equivalent to R_a :*

$$R_t \simeq R_a.$$

Since extension and restriction of scalars over weak equivalences produces Quillen equivalences by [43, A.1.1] and cellularization preserves Quillen equivalences by Corollary A.7, we then have the following corollary.

Corollary 15.2. *There is a zig-zag of Quillen equivalences between the cellularizations of the associated model categories of modules; that is, $\text{cell-}R_t\text{-mod-spectra}$ and $\text{cell-}R_a\text{-mod}$ are Quillen equivalent.*

Proof: We construct another DGA R'_t suitable for comparison and homology isomorphisms $R_a \longrightarrow R''_t \longleftarrow R_t$.

We start at the point $(G/1, c)_{G/1}$, and use localization and splitting to fill in the rest of the diagram. For the moment we work with the $G/1$ row, and omit subscripts, writing $\mathcal{O}_{\mathcal{F}}^t := R_t(G/1, c)$ for brevity. Since

$$H_*(\mathcal{O}_{\mathcal{F}}^t) = \mathcal{O}_{\mathcal{F}} = \prod_F H^*(BG/F),$$

for each finite subgroup F , we may choose a representative cycle \tilde{e}_F for the idempotent corresponding to the F th factor. We may then construct the map

$$\mathcal{O}_{\mathcal{F}}^t \longrightarrow \mathcal{O}_{\mathcal{F}}^t[1/\tilde{e}_F]$$

of DGAs, realizing projection to the F th factor. Choosing representative cycles for the polynomial generators of $H^*(BG/F)$, we obtain a homology isomorphism

$$H^*(BG/F) \longrightarrow \mathcal{O}_{\mathcal{F}}^t[1/\tilde{e}_F].$$

Taking these together we have homology isomorphisms

$$R_t(G/1, c) = \mathcal{O}_{\mathcal{F}}^t \xrightarrow{\simeq} \prod_F \mathcal{O}_{\mathcal{F}}^t[1/\tilde{e}_F] \xleftarrow{\simeq} \prod_F H^*(BG/F) = R_a(G/1, c),$$

so we take

$$R'_t(G/1, c) := \prod_F \mathcal{O}_{\mathcal{F}}^t[1/\tilde{e}_F].$$

Now we pick cycle representatives $\tilde{e}(V)$ in $R_t(G/1, c)$ for Euler classes $e(V)$, and let

$$\tilde{\mathcal{E}}_H = \{\tilde{e}(V), e(V) \mid V^H = 0\}.$$

We then take

$$R''_t(G/H, c) := \tilde{\mathcal{E}}_H^{-1} \prod_F \mathcal{O}_{\mathcal{F}}^t[1/\tilde{e}_F];$$

we will form $R'_t(G/H, c)$ by inverting some more cocycle representatives for elements already invertible in homology. Since the collection $\tilde{\mathcal{E}}_H$ contains Euler classes from both R_a and R_t , we obtain comparison maps

$$(\mathcal{E}_H^t)^{-1} \mathcal{O}_{\mathcal{F}}^t \longrightarrow \tilde{\mathcal{E}}_H^{-1} \prod_F \mathcal{O}_{\mathcal{F}}^t[1/\tilde{e}_F] \longleftarrow \mathcal{E}_H^{-1} \mathcal{O}_{\mathcal{F}}$$

which are homology isomorphisms because the two sets of Euler classes have the same effect in homology. This gives the diagram

$$\begin{array}{ccccccc} R_t(G/1, c) & \longrightarrow & R_t(G/H_1, 1) & \longrightarrow & R_t(G/H_2, 1) & \longrightarrow & \cdots \longrightarrow R_t(G/G, 1) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ R''_t(G/1, c) & \longrightarrow & R''_t(G/H_1, 1) & \longrightarrow & R''_t(G/H_2, 1) & \longrightarrow & \cdots \longrightarrow R''_t(G/G, 1) \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\ \mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}_{H_1}^{-1} \mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}_{H_2}^{-1} \mathcal{O}_{\mathcal{F}} & \longrightarrow & \cdots \longrightarrow \mathcal{E}_G^{-1} \mathcal{O}_{\mathcal{F}} \end{array}$$

We now work up the $\mathbf{LI}(G)$ diagram row by row. When we deal with the G/K row, we assume that the rows G/L with $\dim L < \dim K$ have already been treated. However, if $L \subseteq K$ we will invert some more cycles in the $R''_t(G/L, c)_{G/M}$, to ensure we can compare the G/K row to the G/L row; all these additional elements represent homology classes which are already invertible, so this does not affect the quasi-isomorphisms we have already constructed.

Returning to the G/K row, as before, we take

$$\mathcal{O}_{\mathcal{F}/K}^t := R_t(G/K, c)_{G/K}$$

and choose cocycle representatives $\tilde{e}_{\tilde{K}}$ for idempotents corresponding to the subgroups \tilde{K} with identity component K . We may then form

$$R''_t(G/K, c)_{G/K} := \prod_{\tilde{K} \in \mathcal{F}/K} \mathcal{O}_{\mathcal{F}/K}^t[1/\tilde{e}_{\tilde{K}}].$$

To ensure that this maps to $R'_t(G/L, c)_{G/L}$ we need to invert the images of the cocycle representatives of idempotents from the G/K row under the splitting maps. Similarly, when we invert cocycle representatives of Euler classes to define $R''_t(G/H, c)_{G/K}$ we need to invert their images in $R'_t(G/H, c)_{G/L}$ under the splitting maps. Once this is done, we obtain homology isomorphisms of commutative DGAs

$$R_t \xrightarrow{\simeq} R''_t \xleftarrow{\simeq} R_a$$

on the G/K row of $\mathbf{LI}(G)$ and on all rows G/L with $\dim L < \dim K$. Continuing, row by row we eventually obtain the diagram on all of $\mathbf{LI}(G)$ as required. \square

16. MODEL STRUCTURES AND EQUIVALENCES ON THE ALGEBRAIC CATEGORIES

16.A. Overview and examples. The output of the work above is a Quillen equivalence between the category of rational G -spectra and an algebraic category $\text{cell-}R_a\text{-mod}$, the cellularization of the category of modules over the diagram R_a of rings. The purpose of the next few sections is to simplify the result by avoiding the need for cellularization. In effect, we replace the model by the much smaller category of DG objects in the category of qce-modules, $\mathcal{A}(G)$.

The first example is explained in more detail in [23], and corresponds to the free G -spectra, and it illustrates several of the issues we must deal with.

Example 16.1. The initial model category $\text{cell-}H^*(BG)\text{-mod}_p$ is the category of DG-modules over $H^*(BG)$ with the algebraically projective model structure cellularized with respect to the residue field \mathbb{Q} . This in turn is Quillen equivalent to the model category $\text{cell-}H^*(BG)\text{-mod}_i$, which has the same underlying category of DG-modules over $H^*(BG)$, but now endowed with the algebraically injective model structure and cellularized with respect to the residue field \mathbb{Q} . Finally, this is equivalent to the category $\text{tors-}H^*(BG)\text{-mod}$ of torsion modules with an injective model structure, in which weak equivalences are homology isomorphisms.

The second example corresponds to semi-free G -spectra with $G = S^1$: this is instructive, since one can see the issues introduced by diagrams without having the infinite number of subgroups to complicate matters.

Example 16.2. The diagram of rings is

$$R = \left(\begin{array}{ccc} & R^v & \\ & \downarrow & \\ R^n & \longrightarrow & R^t \end{array} \right) = \left(\begin{array}{ccc} & \mathbb{Q} & \\ & \downarrow & \\ \mathbb{Q}[c] & \longrightarrow & \mathbb{Q}[c, c^{-1}] \end{array} \right)$$

An R_a module M consists of a diagram

$$M = \left(\begin{array}{ccc} & M^v & \\ & \downarrow & \\ M^n & \longrightarrow & M^t \end{array} \right) = \left(\begin{array}{ccc} & V & \\ & \downarrow & \\ N & \longrightarrow & P \end{array} \right)$$

There are four relevant model categories. To start with, on each of the three objectwise module categories we can choose either the algebraically projective model structure or the algebraically injective model structure. We need to make the same choice at each vertex

so that the maps respect the model structures. Secondly, having made that choice we may choose either the diagram theoretically projective or injective model. Assuming the appropriate models all exist, there are Quillen equivalences between either of the two binary choices by using the identity functors. We only need three of the four possibilities; a diagram-projective, algebraically-injective model structure does not appear.

Having made a choice, we cellularize with respect to the two modules corresponding to basic geometric generators

$$\mathbb{S} = R = \left(\begin{array}{ccc} & \mathbb{Q} & \\ & \downarrow & \\ \mathbb{Q}[c] & \longrightarrow & \mathbb{Q}[c, c^{-1}] \end{array} \right) \text{ and } G_+ = \left(\begin{array}{ccc} & 0 & \\ & \downarrow & \\ \mathbb{Q} & \longrightarrow & 0 \end{array} \right)$$

Again, by Appendix A, the Quillen equivalences between model categories can be cellularized.

Finally, for $\text{qce-}R\text{-mod}$, the underlying category consists of quasi-coherent extended modules. The quasi-coherence condition is that the horizontal is localization in the sense that

$$M^t \cong M^n[1/c].$$

The extendedness is the condition that the vertical is induction in the sense that

$$M^t \cong \mathbb{Q}[c, c^{-1}] \otimes_k V.$$

The inclusion of these modules has a right adjoint, and we will give the category of qce- modules a model structure so that it is Quillen equivalent to the doubly injective model structure.

In the remainder of this section we turn to the full $\mathbf{LI}(G)$ -diagram R_a of rings, and outline the proof that the cellularization of the category of R_a -modules is equivalent to the category of DG $\text{qce-}R_a$ -modules (which is to say DG objects of $\mathcal{A}(G)$).

16.B. Construction of model structures. We begin by formally introducing the algebraic model structures we use.

These are model structures on diagrams of modules over diagrams of DGAs. For each individual DGA there is an algebraically projective model structure [42], which is constructed from free modules in the obvious way; the proof may be obtained by adapting [27, 2.3]. Similarly, for an individual DGA with 0 differential there is an algebraically injective model structure [23].

Making a choice of algebraically projective or injective model structures at all points in the diagram we may then seek to define a diagram-theoretically projective model structure (in which weak equivalences and fibrations are given pointwise) or a diagram-theoretically injective model structure (in which weak equivalences and cofibrations are given pointwise). The doubly-projective case follows from [44, 6.1], the doubly-injective case is discussed in Appendix C and the diagram-injective, algebraically-projective case is discussed in Appendix B below.

Finally, given any model structure we may attempt to cellularize it with respect to some collection of cells. The idea is that maps out of these cells should detect weak equivalences, and cofibrant objects should be built from these cells. This is made precise in Appendix A below.

16.C. **A model structure on torsion modules.** We consider the category $\mathcal{A}(G) = \text{qce-}R_a\text{-mod}$, and show the associated category of DG objects admits a model structure with homology isomorphisms as the weak equivalences.

Proposition 16.3. *The category $DG - \mathcal{A}(G)$ of DG qce- R_a -modules admits a model structure with weak equivalences the homology isomorphisms and cofibrations which are monomorphisms at each object. The fibrant objects are injective if the differential is forgotten, and fibrations are surjective maps with fibrant kernel.*

Proof: We use the method of [15, Appendix B], where it is shown that one can often construct a model structure using a type of fibrant generation argument provided one has a suitable finiteness of injective dimension.

We have an abelian category $\mathcal{A} = \mathcal{A}(G)$ and we aim to put a model structure on the category of DG objects of \mathcal{A} . We will specify a set \mathcal{BI} of *basic injectives* containing sufficiently many injectives (i.e., any object of \mathcal{A} embeds in a product of basic injectives). An injective I is viewed as an object $K(I)$ of $DG - \mathcal{A}$ with zero differential. The notation is chosen to suggest an Eilenberg-MacLane object (or cosphere). Next, we let $P(I) = \text{fibre}(1 : K(I) \rightarrow K(I))$, with the notation chosen to suggest a path object (or codisc). The set \mathcal{L} of generating fibrations consists of the maps $P(I) \rightarrow K(I)$ for I in \mathcal{BI} . The set \mathcal{M} of generating acyclic fibrations consists of the maps $P(I) \rightarrow 0$ for I in \mathcal{BI} .

We now take **we** to consist of homology isomorphisms, **cof** to be the maps with the left lifting property with respect to \mathcal{M} and **fib** to be the maps with the right lifting property with respect to $(\text{we} \cap \text{cof})$, and prove this forms the model structure of the lemma. We outline the four main steps and then turn to proving they can be completed in our current situation.

Step 1: Show that **cof** consists of objectwise monomorphisms.

Step 2: Show that for any X there is an objectwise monomorphism $\alpha : X \rightarrow P(I)$ for some injective I .

Step 3: Show that the maps $P(I) \rightarrow K(I)$ and $P(I) \rightarrow 0$ in \mathcal{L} and \mathcal{M} respectively are in **fib**.

Note that since any injective is a retract of a product of basic injectives, it follows that $P(I) \rightarrow K(I)$ and $P(I) \rightarrow 0$ are fibrations for any injective I . Since we have chosen \mathcal{BI} to contain enough injectives, one of the factorization axioms follows immediately, since we may factorize $f : X \rightarrow Y$ as

$$X \xrightarrow{\{f, \alpha\}} Y \times P(I) \xrightarrow{\simeq} Y,$$

with α as in Step 2.

Step 4: Prove the second factorization axiom using only fibrations formed from those named in Step 3.

More precisely, given $f : X \rightarrow Y$, we form factorization $X \rightarrow X' \rightarrow Y$ with $X \rightarrow X'$ a homology isomorphism and $X' \rightarrow Y$ a fibration formed by iterated pullback of fibrations $P(I) \rightarrow K(I)$. This is precisely dual to the usual argument attaching cells to make a map of spaces into a weak equivalence, but because the dual of the small object argument does not apply, we use the finiteness of injective dimension of \mathcal{A} to see that only finitely many steps are involved in the process (details below). The map $X \rightarrow X'$ can be made into a cofibration by taking the product of X' with a suitable $P(I)$ as in the proof of the

first factorization argument. We now see using the defining right lifting property that an arbitrary fibration is a retract of such a standard one.

It remains to verify the four steps can be completed. Once again, we follow the pattern from the case of the circle group. We note that for each connected subgroup H of G there is an evaluation functor

$$ev_H : R_a\text{-modules} \longrightarrow \mathcal{O}_{\mathcal{F}/H}\text{-modules}$$

with right adjoint f_H . In particular, if N is a torsion module, $f_H(N)$ lies in $\mathcal{A}(G)$ and

$$\mathrm{Hom}_{\mathcal{A}(G)}(X, f_H(N)) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/H}}(X(H), N).$$

We take the basic injectives to be those of form

$$\mathbb{I}_{\tilde{H}} = f_H(H_*(BG/\tilde{H}))$$

where \tilde{H} is a subgroup with identity component H . It is shown in [18] that this set contains sufficiently many injectives.

The following elementary lemma lets us reduce verifications to statements about modules with zero differential over a (single object) ring.

Lemma 16.4. (1) $\mathrm{Hom}(X, K(f_H(M))) = \mathrm{Hom}(X(H), M)$
(2) $DG - \mathrm{Hom}(X, K(f_H(M))) = \mathrm{Hom}(X(H)/dX(H), M)$
(3) $DG - \mathrm{Hom}(X, P(f_H(M))) = \mathrm{Hom}(\Sigma X(H), M)$

It follows by the left lifting property that **cof** consists of objectwise monomorphisms (Step 1), and that we may find a monomorphism α in the first factorization argument (Step 2). This lemma also makes it straightforward to verify that objects of \mathcal{L} and \mathcal{M} are fibrations. The case of $P(I) \longrightarrow 0$ is simply the defining property of an injective. The problem

$$\begin{array}{ccc} A & \longrightarrow & P(f_H(\mathbb{I}_{\tilde{H}})) \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ B & \longrightarrow & K(f_H(\mathbb{I}_{\tilde{H}})) \end{array}$$

is equivalent to

$$\begin{array}{ccc} A(H) & \longrightarrow & \Sigma^{-1}\mathbb{I}_{\tilde{H}} \\ \downarrow & \nearrow \text{dashed} & \uparrow \\ B(H) & \xleftarrow{d} & \Sigma^{-1}B(H)/dB(H) \end{array}$$

To find a solution, we note that the diagonal is already defined consistently on $A(H) + dB(H)$ and use the defining property of injectives to extend it over $B(H)$.

This leaves Step 4. Here we start by forming an exact sequence

$$0 \longrightarrow H_*(X) \longrightarrow H_*(Y) \oplus I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_N \longrightarrow 0$$

in $\mathcal{A}(G)$, where the I_s are injective. The finite injective dimension of $\mathcal{A}(G)$ ensures such an exact sequence exists. We now realize this by a tower of fibrations

$$Y \longleftarrow X_0 \longleftarrow \cdots \longleftarrow X_N = X',$$

together with lifts

$$\begin{array}{ccc}
 & & \downarrow \\
 & & X_1 \\
 & \nearrow f_1 & \downarrow \\
 & & X_0 \\
 & \nearrow f_0 & \downarrow \\
 X & \xrightarrow{f} & Y.
 \end{array}$$

We take $X_0 = Y \oplus K(I_0)$, and the subsequent objects and maps are constructed using the diagram

$$\begin{array}{ccc}
 X & & \\
 \searrow & & \searrow \\
 & X_s & \longrightarrow P(\Sigma^{-s} I_s) \\
 \searrow & \downarrow & \downarrow \\
 & X_{s-1} & \longrightarrow K(\Sigma^{-s} I_s)
 \end{array}$$

where the lower horizontal is chosen to realize the inclusion of $\text{im}(I_{s-1} \rightarrow I_s)$ in I_s . The map $f_N : X \rightarrow X_N$ is necessarily a homology isomorphism, and can be made into a monomorphism by taking a product with a suitable $P(I)$.

This completes the sketch proof of the proposition. \square

16.D. Equivalence of models of torsion modules. In this subsection, we consider the category $\mathcal{A}(G)$ of qce modules over the diagram of rings R_a . It was proved in [19] that there is an adjunction with i the inclusion of the subcategory

$$i : \text{qce-}R_a\text{-mod} \rightleftarrows R_a\text{-mod} : \Gamma .$$

We use the doubly injective model structure on R_a -modules (i.e., injective in both the module theoretic and diagram theoretic sense).

Proposition 16.5. *The adjunction*

$$i : \text{qce-}R_a\text{-mod} \rightleftarrows R_a\text{-mod}_{ii} : \Gamma .$$

is a Quillen adjunction. Cellularizing with respect to the images of the topological cells induces a Quillen equivalence

$$\mathcal{A}(G) = \text{qce-}R_a\text{-mod} \simeq \text{cell-}R_a\text{-mod}_{ii}.$$

Proof: First, i preserves all homology isomorphisms, so to see it is a left Quillen functor we need only check that it preserves cofibrations. The cofibrations of $\mathcal{A}(G)$ are objectwise monomorphisms. Next note that the cofibrations in the algebraically injective model structure are precisely the monomorphisms. The cofibrations in the doubly injective R_a -module category are precisely the morphisms which are objectwise cofibrations, namely the objectwise monomorphisms. Thus i preserves cofibrations.

We may then cellularize with respect to the images of the cells G/H_+ . By the Cellularization Principle A.6 this induces a Quillen equivalence since the cells are already torsion modules by Lemma 17.6.

$$\text{cell-qce-}R_a\text{-mod} \simeq \text{cell-}R_a\text{-mod}_{ii}$$

Finally, it remains to check that cellularization is the identity on $\mathcal{A}(G)$. This will be completed by Proposition 17.8 which states that cellular equivalences for qce modules are precisely the homology isomorphisms. Thus,

$$\mathcal{A}(G) = \text{qce-}R_a\text{-mod} = \text{cell-qce-}R_a\text{-mod}.$$

□

17. CELLULAR EQUIVALENCES IN $\mathcal{A}(G)$

The main purpose of this section is to show that cellular equivalences coincide with homology isomorphisms for $\mathcal{A}(G)$. This comes in Subsection 17.E. Since cells are determined by their homology we need not choose particular models. Nonetheless, we begin by describing some models for the algebraic cells, since this gives us an opportunity to introduce some essential properties in a concrete fashion.

17.A. Cohomology of subgroups and homotopy of cells. To guide our construction, we calculate the homology of the natural cells G/K_+ as R_a -modules, and then give the appropriate adaption using Koszul complex constructions. First we need a little more background. We have already discussed the relationship between G and its quotient groups G/K , together with the associated inflation map $\mathcal{O}_{\mathcal{F}/K} \rightarrow \mathcal{O}_{\mathcal{F}}$. We now need to discuss the cohomology of subgroups and the associated restriction maps.

To start with note that rational cohomology of a subgroup K of the torus depends only on the identity component: the restriction map induces an isomorphism $H^*(BK) \xrightarrow{\cong} H^*(BK_1)$, since the component group K/K_1 necessarily acts trivially on $H^*(BK_1)$.

Next, write $\mathcal{F}(K)$ for the set of finite subgroups of K when emphasis is required, and similarly

$$\mathcal{O}_{\mathcal{F}}^K = \prod_{F \in \mathcal{F}(K)} H^*(BK/F),$$

noting that this makes sense whether or not K is connected. The restriction map is the composite

$$\mathcal{O}_{\mathcal{F}}^G = \prod_{F \in \mathcal{F}(G)} H^*(BG/F) \longrightarrow \prod_{F \in \mathcal{F}(K)} H^*(BG/F) \longrightarrow \prod_{F \in \mathcal{F}(K)} H^*(BK/F) = \mathcal{O}_{\mathcal{F}}^K$$

where the first map is projection onto the direct summand corresponding to the factors with $F \in \mathcal{F}(K)$.

Lemma 17.1. *If K is a subgroup of G of codimension $c(K)$ then as an R_a -module $\pi_*^{\mathcal{A}}(G/K_+)$ is concentrated on the connected subgroups of K and is given by*

$$\pi_*^{\mathcal{A}}(G/K_+)(L) = \begin{cases} \mathcal{E}_L^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/L}} \Sigma^{c(K)} \mathcal{O}_{\mathcal{F}/L}^{\overline{K}} & \text{if } L \text{ is a connected subgroup of } K \\ 0 & \text{otherwise.} \end{cases}$$

where bars indicate images in $\overline{G} = G/L$.

Proof: We must calculate

$$\pi_*^G(DEF_+ \wedge S^{\infty V(L)} \wedge G/K_+) = \mathcal{E}_L^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/L}} \pi_*^{G/L}(DEF/L_+ \wedge \Phi^L G/K_+).$$

This is evidently zero unless $L \subseteq K$, and if $L \subseteq K$, then $\Phi^L G/K_+ = (G/K)_+^L = \overline{G}/\overline{K}_+$, where bars indicate the image in G/L . Accordingly,

$$\pi_*^{\overline{G}}(DEF/L_+ \wedge \overline{G}/\overline{K}_+) \cong \Sigma^{c(K)} \pi_*^{\overline{K}}(DEF_+/L) = \Sigma^{c(K)} \prod_{L \subseteq \tilde{L} \subseteq K} H^*(B(K/\tilde{L})).$$

□

17.B. Koszul complexes. To form suitable models, we use a standard construction from commutative algebra. Given a graded commutative ring B and elements x_1, \dots, x_n we may form the Koszul complex

$$K(x_1, \dots, x_n) = (\Sigma^{|x_1|} B \xrightarrow{x_1} B) \otimes_B \cdots \otimes_B (\Sigma^{|x_n|} B \xrightarrow{x_n} B),$$

which is finitely generated and free as a B -module. If B is a polynomial ring $B = k[x_1, \dots, x_n]$ we write just Kos_B for the complex; this is independent of the choice of homogeneous generators up to isomorphism, and the natural map $\text{Kos}_B \rightarrow k$ is a homology isomorphism. If M is a B -module we write $\text{Kos}_B(M) = \text{Kos}_B \otimes_B M$.

We say that L is *cotoral* in K if L is a normal subgroup of K and K/L is a torus. We note that if L is cotoral in K then we may choose a map $G/L \rightarrow K/L$ giving an isomorphism $G/L = G/K \times K/L$ and hence

$$H^*(BG/L) = H^*(BG/K) \otimes H^*(BK/L).$$

We may therefore form a version of $H^*(BG/K)$ which is flat over $H^*(BG/L)$ by tensoring with the Koszul complex model for $H^*(BK/L)$ based on a set of generators for $\ker(H^*(BG/L) \rightarrow H^*(BK/L))$. However it is not necessary to make this choice, since (even if K/L is not a torus), the exact sequence

$$K/L \rightarrow G/L \rightarrow G/K$$

induces an exact sequence

$$H^*(BK/L) \leftarrow H^*(BG/L) \leftarrow H^*(BG/K)$$

of Hopf algebras, so that

$$H^*(BK/L) = H^*(BG/L) \otimes_{H^*(BG/K)} k.$$

Now replace k by the complex $\text{Kos}_{H^*(BG/K)}$ of projective $H^*(BG/K)$ -modules, independent of L , and the tensor product will be a complex of projective $H^*(BG/L)$ -modules,

$$\text{Kos}_{H^*(BG/K)}(H^*(BG/L)) \simeq H^*(BK/L).$$

If $H^*(BG/K)$ is replaced by a DGA $A(K)$ with the same cohomology, the complexes $\Sigma^{|x|} B \rightarrow B$ are replaced by the fibres of the maps $\Sigma^{|x|} A(K) \rightarrow A(K)$; up to equivalence, this only depends on the cohomology classes. The full Koszul complex $\text{Kos}_{A(K)}$ is obtained as before by tensoring together the DG modules for a chosen set of polynomial generators.

17.C. The flat form of the natural cells. To apply the Koszul complexes in our case we choose generators for $H^*(BG/K)$ and form the associated Koszul complex, $\text{Kos}_{H^*(BG/K)}$. Note that we have a diagonal map

$$\Delta : H^*(BG/K) \longrightarrow \prod_{\tilde{L} \in \mathcal{F}/L} H^*(BG/\tilde{L}) \cong \prod_{\tilde{L} \in \mathcal{F}/L} H^*(BG/\tilde{L}) = \mathcal{O}_{\mathcal{F}/L}^G,$$

so we may form $\text{Kos}_{H^*(BG/K)}(\mathcal{O}_{\mathcal{F}/L}^G)$, which is free as a $\mathcal{O}_{\mathcal{F}/L}^G$ -module.

Lemma 17.2. *The Koszul complex gives good approximations of quotient groups in the sense that*

$$H_*(e_{\overline{K}} \mathcal{O}_{\mathcal{F}/L}^{G/L} \otimes_{\mathcal{O}_{\mathcal{F}/L}^G} \text{Kos}_{H^*(BG/K)}(\mathcal{O}_{\mathcal{F}/L}^G)) \cong \mathcal{O}_{\mathcal{F}/L}^{\overline{K}},$$

where $e_{\overline{K}}$ is the idempotent corresponding to the finite subgroups of G/L contained in \overline{K} .

Proof: We have remarked that the short exact sequence

$$0 \longrightarrow H^*(BG/K) \longrightarrow H^*(BG/L) \longrightarrow H^*(BK/L) \longrightarrow 0$$

of Hopf algebras shows that

$$H_*(H^*(BG/L) \otimes_{H^*(BG/K)} \text{Kos}_{H^*(BG/K)}) \cong H^*(BK/L).$$

We are just taking a product of instances of this indexed by finite subgroups of G/L contained in K/L . \square

This motivates the following definition.

Definition 17.3. The flat form of model for the cell G/K_+ is defined by

$$\underline{\sigma}_K(L) = \begin{cases} \mathcal{E}_L^{-1} e_K \mathcal{O}_{\mathcal{F}}^G \otimes_{\mathcal{O}_{\mathcal{F}/K}} \Sigma^{c(K)} \text{Kos}_{H^*(BG/K)}(\mathcal{O}_{\mathcal{F}/K}) & \text{if } L \text{ is cotoral in } K_1 \\ 0 & \text{otherwise} \end{cases}$$

We may now prove that this is indeed a model.

Lemma 17.4. *The flat model $\underline{\sigma}_K$ is a model for G/K_+ in $\mathcal{A}(G)$ and any other model is weakly equivalent to it.*

Proof: The uniqueness theorem [18, 12.1] states that cells are characterized by their homology, so it suffices to show that

$$H_*(\underline{\sigma}_K) \cong \pi_*^{\mathcal{A}}(G/K_+).$$

Lemmas 17.1 and 17.2 show that the value at L should be

$$\mathcal{E}_L^{-1} \mathcal{O}_{\mathcal{F}}^G \otimes_{\mathcal{O}_{\mathcal{F}/L}} e_{\overline{K}} \mathcal{O}_{\mathcal{F}/L}^{\overline{G}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \Sigma^{c(K)} \text{Kos}_{H^*(BG/K)}(\mathcal{O}_{\mathcal{F}/K}),$$

and we calculate

$$\mathcal{E}_L^{-1} \mathcal{O}_{\mathcal{F}}^G \otimes_{\mathcal{O}_{\mathcal{F}/L}} e_{\overline{K}} \mathcal{O}_{\mathcal{F}/L}^{\overline{G}} \cong \mathcal{E}_L^{-1} e_K \mathcal{O}_{\mathcal{F}}^G.$$

\square

It is worth recording the following immediate consequence of the definition.

Lemma 17.5. *The flat model $\underline{\sigma}_K$ is built from the model $\underline{\sigma}_G = \mathcal{O}_{\mathcal{F}}^G$ of the sphere by taking a retract and then using finitely many fibre sequences.* \square

17.D. **Properties of the flat model $\underline{\sigma}_K$.** By construction the cells themselves have torsion homology, which gives one of the properties we require.

Lemma 17.6. *In the model category $\text{cell-}R_a\text{-mod}$ the cellular objects have torsion homology.*

Proof: By construction, $H_*(\sigma_K) = \pi_*^{\mathcal{A}}(G/K_+)$, which lies in $\mathcal{A}(G)$. Since the subcategory of torsion modules is an abelian subcategory closed under sums, any object built from cells has torsion homology. \square

Next, there is a finiteness requirement if we are to use these as the generating objects for our cofibrantly generated model structure.

Lemma 17.7. *For any subgroup K , the flat model σ_K for the cell G/K_+ is compact in the sense that*

$$\text{Hom}(\sigma_K, \bigoplus_i N_i) = \bigoplus_i \text{Hom}(\sigma_K, N_i).$$

Proof: In view of Lemma 17.5, it suffices to prove the special case $K = G$.

The value of a map $f : \sigma_G \rightarrow M$ is determined by its value at $L = 1$. On the other hand

$$f(1) : \sigma_G(1) = \mathcal{O}_{\mathcal{F}} \rightarrow M(1)$$

is determined by the image of the identity. \square

17.E. **Algebraic cells and homology isomorphisms.** We need to understand weak equivalences between objects of $\mathcal{A}(G)$. For clarity, we refer to maps $X \rightarrow Y$ inducing an isomorphism of $H_*(\text{Hom}(\underline{\sigma}_K, \cdot))$ for all cells σ_K as *cellular equivalences*, and for brevity we write

$$\pi_*^G(X) = H_*(\text{Hom}(\underline{\sigma}_\bullet, \cdot))$$

for the resulting Mackey-functor.

We begin with a warning: the flat models are not cofibrant. For example if G is a circle, we have

$$\text{Hom}(S^0, X) = PB(X),$$

where X and its pullback are as displayed:

$$\begin{array}{ccc} PB(X) & \longrightarrow & V \\ \downarrow & & \downarrow \\ N & \longrightarrow & \mathcal{E}_G^{-1} \mathcal{O}_{\mathcal{F}} \otimes V. \end{array}$$

It is not hard to construct examples of acyclic X for which $PB(X)$ is not acyclic. This simply means that we retreat from being so explicit at the level of models. Our remaining work takes place at the level of homotopy categories.

The key to removing the cellularization process in the formation of $\mathcal{A}(G)$ is to show that there are enough cells in the sense that cellular equivalences of torsion modules are homology isomorphisms.

Proposition 17.8. *For objects of $\mathcal{A}(G)$ cellular equivalences are homology isomorphisms.*

Proof: Taking mapping cones, it suffices to show that cellularly trivial objects of $\mathcal{A}(G)$ are acyclic. Suppose then that X is cellularly trivial.

We shall mimic the following argument for spectra. Since S^0 is built from objects $E(\mathcal{F}/K)_+ \wedge S^{\infty V(K)}$ by finitely many cofibre sequences, X is built from $X \langle \mathcal{F}/K \rangle = X \wedge E(\mathcal{F}/K)_+ \wedge S^{\infty V(K)}$ by finitely many cofibre sequences, and it suffices to show that the homology of $X \langle \mathcal{F}/K \rangle$ is zero. However, for spectra of this form, cellular triviality and acyclicity are equivalent. Finally, by duality of cells, cellular triviality of X implies that of $X \wedge T$ for any T .

Before we begin, we need two lemmas.

Lemma 17.9. *If $\pi_*^G(X) = 0$ then $\pi_*^G(X \otimes C) = 0$ for any cellular object C .*

Proof: The lemma follows from the special case in which C is a cell $\underline{\sigma}_L$. In fact, we have a homotopy equivalence of DGAs

$$\mathrm{Hom}(\underline{\sigma}_K, X \otimes \underline{\sigma}_L) \cong \Sigma^{c(L)} \mathrm{Hom}(\underline{\sigma}_{K \cap L}, X) \otimes H^*(G/(KL))$$

where $H^*(G/(KL))$ has zero differential and $c(L)$ is the codimension of L .

In fact both sides can be formed from $e_{K \cap L} \mathrm{Hom}(S^0, X)$ by taking iterated fibres of maps which are multiplication by some element of $\mathcal{O}_{\mathcal{F}}$ obtained by inflation. On the left we take a set of polynomial generators for $H^*(BG/K)$ and a set y_1, \dots, y_e of polynomial generators for $H^*(BG/L)$ and inflate them. It is convenient to choose generators x_1, \dots, x_d of $H^*(BG/(KL))$ and extend the collection by y_1, \dots, y_e to give generators of $H^*(BG/K)$ and by z_1, \dots, z_f to give generators of $H^*(BG/L)$. On the left we may then choose $x_1, \dots, x_d, y_1, \dots, y_e, z_1, \dots, z_f$ as our generators of $H^*(BG/(K \cap L))$. Since the map x of a Koszul complex $K(x)$ is nullhomotopic, the effect of the second set of generators x_1, \dots, x_e simply increases multiplicity. \square

We need to use certain standard objects of $\mathcal{A}(G)$ constructed from modules. There is a right adjoint f_K to evaluation at G/K . Roughly speaking, for suitable $\mathcal{O}_{\mathcal{F}/K}$ -modules M , the object $f_K(M)$ is obtained by putting M at G/K , and filling in other values accordingly (see [18, Section 4] for further details).

Lemma 17.10. *Suppose X is a torsion object with $H_*(X) = f_K(M)$ for some connected subgroup K and some $\mathcal{O}_{\mathcal{F}/K}$ -module M . If X is cellularly trivial it is acyclic: if $\pi_*^G(X) = 0$ then $H_*(X) = 0$.*

Proof: Since f_K is right adjoint to evaluation at K , and since this is compatible with resolutions, the Adams spectral sequence for $[T, X]^G$ takes the simple form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{O}_{\mathcal{F}/K}}^{*,*}(\phi^K H_*(T), M) \Rightarrow [T, X]_*^G.$$

In particular, taking $T = S^0$, we see that $H_*(X)$ is one of the entries in $\pi_*^G(X)$. \square

We may now proceed to prove the proposition. We argue by induction on the dimension of support of $H_*(X)$ that cellularly trivial objects of $\mathcal{A}(G)$ are acyclic. There is nothing to prove if the support is in dimension < 0 (i.e., if $H_*(X) = 0$).

Suppose then that the result is proved for objects with support in dimension $< d$ and that X is supported in dimension $\leq d$. We then define X' using the fibre sequence

$$X' \longrightarrow X \longrightarrow \bigoplus_{\dim(K)=d} f_K(\phi^K H_*(X)).$$

Next, note that

$$f_K(\phi^K H_*(X)) \simeq X \otimes S^{\infty V(K)}.$$

This is cellularly trivial by Lemma 17.9, and hence also acyclic by Lemma 17.10.

Finally, we claim there is an equivalence

$$X' \simeq X \otimes F$$

for suitable F . Indeed, since X is supported in dimension $\leq d$ we have $X \simeq X \otimes E\mathcal{F}(d)_+$, where $\mathcal{F}(d)$ is the family of subgroups of dimension $\leq d$. We may then take F to be defined by the fibre sequence

$$F \longrightarrow E\mathcal{F}(d)_+ \longrightarrow \bigvee_{\dim(K)=d} E\mathcal{F}(d)_+ \otimes S^{\infty V(K)}.$$

Thus X' is cellularly trivial by Lemma 17.9 and it is thus acyclic by induction. It follows that X is acyclic, which completes the inductive step. The general case follows in $r+1$ steps. \square

APPENDIX A. CELLULARIZATION OF MODEL CATEGORIES

Throughout the paper we need to consider models for categories of cellular objects, thought of as constructed from a set of basic cells using coproducts and cofibre sequences. These models are usually obtained by the process of cellularization (sometimes known as right localization or colocalization) of model categories, with the cellular objects appearing as the cofibrant objects. Because it is fundamental to our work, we recall some of the basic definitions from [26].

Definition A.1. [26, 3.1.8] Let \mathbb{M} be a model category and \mathcal{K} be a set of objects in \mathbb{M} . A map $f : X \rightarrow Y$ is a \mathcal{K} -cellular equivalence if for every element A in \mathcal{K} the induced map of homotopy function complexes [26, 17.4.2] $f_* : \text{map}(A, X) \rightarrow \text{map}(A, Y)$ is a weak equivalence. An object W is \mathcal{K} -cellular if W is cofibrant in \mathbb{M} and $f_* : \text{map}(W, X) \rightarrow \text{map}(W, Y)$ is a weak equivalence for any \mathcal{K} -cellular equivalence f .

One can cellularize a right proper model category under very mild finiteness hypotheses. To minimize the confusion due to a coincidence of nomenclature, we refer to a model category satisfying [26, 12.1.1] as H -cellular.

Proposition A.2. [26, 5.1.1] *Let \mathbb{M} be a right proper H -cellular model category and let \mathcal{K} be a set of objects in \mathbb{M} . The \mathcal{K} -cellularized model category $\mathcal{K}\text{-cell-}\mathbb{M}$ exists and has weak equivalences the \mathcal{K} -cellular equivalences, fibrations the fibrations in \mathbb{M} and cofibrations the maps with the left lifting property with respect to the trivial fibrations. The cofibrant objects are the \mathcal{K} -cellular objects.*

Remark A.3. Since the \mathcal{K} -cellular equivalences are defined using homotopy function complexes, the \mathcal{K} -cellularized model category $\mathcal{K}\text{-cell-}\mathbb{M}$ depends only on the homotopy type of the objects in \mathcal{K} .

It is useful to have the following further characterization of the cofibrant objects.

Proposition A.4. [26, 5.1.5] *If \mathcal{K} is a set of cofibrant objects in \mathbb{M} , then the class of \mathcal{K} -cellular objects agrees with the smallest class of cofibrant objects in \mathbb{M} that contains \mathcal{K} and is closed under homotopy colimits and weak equivalences.*

Since we are always working with stable model categories here, homotopy classes of maps out of \mathcal{K} detect trivial objects. That is, in $\text{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$, an object X is trivial if and only if for each element A in \mathcal{K} , $[A, X]_* = 0$. In this case, by [43, 2.2.1] we have the following.

Proposition A.5. *If \mathbb{M} is stable and each element of \mathcal{K} is compact, then \mathcal{K} is a set of generators of $\text{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$. That is, the only localizing subcategory containing \mathcal{K} is $\text{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$ itself.*

We next need to show that appropriate cellularizations of these model categories preserve Quillen adjunctions and induce Quillen equivalences.

Proposition A.6. The Cellularization Principle. *Let \mathbb{M} and \mathbb{N} be right proper, stable, H-cellular model categories with $F : \mathbb{M} \rightarrow \mathbb{N}$ a Quillen adjunction with right adjoint U . Let Q be a cofibrant replacement functor in \mathbb{M} and R a fibrant replacement functor in \mathbb{N} .*

- (1) *Let $\mathcal{K} = \{A_\alpha\}$ be a set of compact objects in \mathbb{M} , with $FQ\mathcal{K} = \{FQA_\alpha\}$ the corresponding set of objects in \mathbb{N} . Then F and U induce a Quillen adjunction between the \mathcal{K} -cellularization of \mathbb{M} and the $FQ\mathcal{K}$ -cellularization of \mathbb{N} .*

$$F : \mathcal{K}\text{-cell-}\mathbb{M} \rightleftarrows FQ\mathcal{K}\text{-cell-}\mathbb{N} : U$$

- (2) *Furthermore, if $QA \rightarrow URFQA$ is a weak equivalence in \mathbb{M} for each A in \mathcal{K} , then F and U induce a Quillen equivalence.*

$$\mathcal{K}\text{-cell-}\mathbb{M} \simeq_Q FQ\mathcal{K}\text{-cell-}\mathbb{N}$$

- (3) *Let $\mathcal{J} = \{B_\beta\}$ be a set of compact objects in \mathbb{N} , with $UR\mathcal{J} = \{URB_\beta\}$ the corresponding set of objects in \mathbb{M} . If $FQURB \rightarrow RB$ is a weak equivalence in \mathbb{N} for each B in \mathcal{J} , then F and U induce a Quillen equivalence between the \mathcal{J} -cellularization of \mathbb{N} and the $UR\mathcal{J}$ -cellularization of \mathbb{M} .*

$$UR\mathcal{J}\text{-cell-}\mathbb{M} \simeq_Q \mathcal{J}\text{-cell-}\mathbb{N}$$

Proof: The criterion in [26, 3.3.18(2)] (see also [28, 2.2]) for showing that F and U induce a Quillen adjoint pair on the cellularized model categories requires that U preserves weak equivalences. Any Quillen adjunction induces a weak equivalence $\text{map}(A, URX) \simeq \text{map}(FQA, X)$ of the homotopy function complexes, see for example [26, 17.4.15]. So a map $f : X \rightarrow Y$ induces a weak equivalence $f_* : \text{map}(FQA, X) \rightarrow \text{map}(FQA, Y)$ if and only if $Uf_* : \text{map}(A, UX) \rightarrow \text{map}(A, UY)$ is a weak equivalence. Thus in (1), U preserves (and reflects) the cellular equivalences. Similarly, $Uf_* : \text{map}(URB, UX) \rightarrow \text{map}(URB, UY)$ is a weak equivalence if and only if $f_* : \text{map}(FQURB, X) \rightarrow \text{map}(FQURB, Y)$ is. Given that $FQURB \rightarrow RB$ is a weak equivalence, it follows that Uf_* is a weak equivalence if and only

if $f_* : \text{map}(B, X) \rightarrow \text{map}(B, Y)$ is. Thus, given the hypothesis in (3) it follows that U preserves (and reflects) the cellular equivalences.

Next we establish the Quillen equivalence in (2); the arguments are very similar for (3). Since \mathbb{M} and \mathbb{N} are stable, it follows that the cellularizations are also stable. The Quillen adjunction in (1) induces a derived adjunction on the triangulated homotopy categories; we show that this is actually a derived equivalence. We consider the full subcategories of objects M in $\text{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$ and N in $\text{Ho}(FQ\mathcal{K}\text{-cell-}\mathbb{N})$ such that the unit $QM \rightarrow URFQM$ or counit $FQURN \rightarrow RN$ of the adjunctions are equivalences. Since both derived functors are exact and preserve coproducts, these are localizing subcategories. Since for each A in \mathcal{K} the unit is an equivalence and \mathcal{K} is a set of generators by Proposition A.5, the unit is an equivalence on all of $\text{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$. It follows that the counit is also an equivalence for each object $N = FQA$ in $FQ\mathcal{K}$. Since $FQ\mathcal{K}$ is a set of generators for $\text{Ho}(FQ\mathcal{K}\text{-cell-}\mathbb{N})$, the counit is also always an equivalence. Statement (2) follows. \square

Note that if F and U form a Quillen equivalence on the original categories, then the conditions in Proposition A.6 parts (2) and (3) are automatically satisfied. Thus, they also induce Quillen equivalences on the cellularizations.

Corollary A.7. *Let \mathbb{M} and \mathbb{N} be right proper H-cellular model categories with $F : \mathbb{M} \rightarrow \mathbb{N}$ a Quillen equivalence with right adjoint U . Let Q be a cofibrant replacement functor in \mathbb{M} and R a fibrant replacement functor in \mathbb{N} .*

- (1) *Let $\mathcal{K} = \{A_\alpha\}$ be a set of compact objects in \mathbb{M} , with $FQ\mathcal{K} = \{FQA_\alpha\}$ the corresponding set of objects in \mathbb{N} . Then F and U induce a Quillen equivalence between the \mathcal{K} -cellularization of \mathbb{M} and the $FQ\mathcal{K}$ -cellularization of \mathbb{N} :*

$$\mathcal{K}\text{-cell-}\mathbb{M} \simeq_Q FQ\mathcal{K}\text{-cell-}\mathbb{N}$$

- (2) *Let $\mathcal{J} = \{B_\beta\}$ be a set of compact objects in \mathbb{N} , with $UR\mathcal{J} = \{URB_\beta\}$ the corresponding set of objects in \mathbb{N} . Then F and U induce a Quillen equivalence between the \mathcal{J} -cellularization of \mathbb{N} and the $UR\mathcal{J}$ -cellularization of \mathbb{M} :*

$$UR\mathcal{J}\text{-cell-}\mathbb{M} \simeq_Q \mathcal{J}\text{-cell-}\mathbb{N}$$

In [28, 2.3] Hovey gives criteria for when localizations preserve Quillen equivalences. Since cellularization is dual to localization, this corollary also follows from the dual of Hovey's statement.

APPENDIX B. DIAGRAM-INJECTIVE MODEL STRUCTURES

In this section we develop *diagram-projective* and *diagram-injective* model structures for a generalized category of diagrams where the category at each spot in the diagram is allowed to vary. As we see in Remark B.7 below, one example of such a generalized category of diagrams is the category of module spectra over a spectral category (ring spectrum with many objects), [43, 3.3.2]. In that case, the *diagram-projective* (or *standard*) model structure agrees with the one developed in [43, A.1.1] and has objectwise weak equivalences and fibrations; see also [1]. In contrast, the diagram-injective model structure here has weak equivalences and cofibrations determined at each object. These are the analogues of the model structures for diagrams over direct and inverse small categories developed, for example, in [27, 5.1.3].

We restrict our attention here to the diagrams indexed on small direct (or inverse) categories. Let \mathbf{D} be a small direct category with a fixed linear extension $d : \mathbf{D}^{op} \rightarrow \lambda$ for some ordinal λ . Note that if $\mathbf{D}(i, j)$ is non-empty then $d(i) < d(j)$. Let \mathbb{M} be a diagram of model categories indexed by \mathbf{D} ; that is, for each $i \in \mathbf{D}$ a model category $\mathbb{M}(i)$ and for each $a : i \rightarrow j$ in \mathbf{D} a left Quillen functor $F_a : \mathbb{M}(i) \rightarrow \mathbb{M}(j)$ (with right adjoint G_a) such that the diagram commutes. Then a diagram X over \mathbb{M} (or “ \mathbb{M} -diagram”) specifies for each object i in \mathbf{D} an object $X(i)$ of $\mathbb{M}(i)$ and for each morphism $a : i \rightarrow j$ in \mathbf{D} a map $X(a) : F_a X(i) \rightarrow X(j)$ such that the diagrams commute. Let \mathbf{D}_i be the category of all non-identity maps with codomain i in \mathbf{D} . Then any diagram X induces a functor \widetilde{X}_i from \mathbf{D}_i to $\mathbb{M}(i)$ by taking $a : j \rightarrow i$ in \mathbf{D}_i to $F_a X(j)$. Define the *latching space* functor, $L_i X$ as the direct limit in $\mathbb{M}(i)$,

$$L_i X = \lim_{\rightarrow \mathbf{D}_i} \widetilde{X}_i.$$

In the dual situation where \mathbf{D} is a small inverse category, we consider the same diagram of model categories \mathbb{M} and the same category of diagrams X over \mathbb{M} . Note, here again each $a : i \rightarrow j$ in \mathbf{D} is assigned to a left Quillen functor $F_a : \mathbb{M}(i) \rightarrow \mathbb{M}(j)$ with right adjoint G_a . Let \mathbf{D}^i be the category of all non-identity maps with domain i in \mathbf{D} . Then any diagram X induces a functor \widetilde{X}^i from \mathbf{D}^i to $\mathbb{M}(i)$ by taking $a : i \rightarrow j$ in \mathbf{D}^i to $G_a X(j)$. Define the *matching space* functor, $M_i X$ as the inverse limit in $\mathbb{M}(i)$ of \widetilde{X}^i .

Theorem B.1. *Assume given a category \mathbf{D} and a diagram of model categories, \mathbb{M} , indexed on \mathbf{D} as above.*

(i) *If \mathbf{D} is a direct category, then there is a diagram-projective model structure on the category of diagrams over \mathbb{M} with objectwise weak equivalences and fibrations; that is, $X \rightarrow Y$ is a weak equivalence (or fibration) if $X(i) \rightarrow Y(i)$ is an underlying weak equivalence (or fibration) in $\mathbb{M}(i)$ for all i . This map is a (trivial) cofibration if and only if the induced map $X(i) \coprod_{L_i X} L_i Y \rightarrow Y(i)$ is a (trivial) cofibration in $\mathbb{M}(i)$ for all i .*

(ii) *If \mathbf{D} is an inverse category, then there is a diagram-injective model structure on the category of diagrams over \mathbb{M} with objectwise weak equivalences and cofibrations; that is, $X \rightarrow Y$ is a weak equivalence (or cofibration) if $X(i) \rightarrow Y(i)$ is an underlying weak equivalence (or cofibration) in $\mathbb{M}(i)$ for all i . This map is a (trivial) fibration if and only if the induced map $X(i) \rightarrow Y(i) \times_{M_i Y} M_i X$ is a (trivial) fibration in $\mathbb{M}(i)$ for all i .*

Proof: The verification of the axioms follows the same outline as in [27, 5.1.3]. The only difference is that here the ambient category changes at each object in \mathbf{D} . Instead of repeating these arguments, we give some of the details for these changing categories. As in [27, 5.1.3] we consider only the diagram-projective or direct category case, since the diagram-injective or inverse category case is dual.

Define $\mathbf{D}_{<\beta}$ as the full subcategory of \mathbf{D} on all objects i such that $d(i) < \beta$. Then let $\mathbb{M}_{<\beta}$ denote the diagram of model categories induced by the restriction of \mathbb{M} to $\mathbf{D}_{<\beta}$. Similarly, for any \mathbb{M} diagram X , the restriction to $\mathbf{D}_{<\beta}$ gives an $\mathbb{M}_{<\beta}$ diagram $X_{<\beta}$. Given these definitions, the lifting axioms follow by induction as in [27, 5.1.4], by producing lifts for the various restrictions to $\mathbb{M}_{<\beta}$ diagrams. Note that at the successor ordinal case the relevant commutative diagram is just a usual lifting problem in $\mathbb{M}(i)$.

To complete the verification of the model structure we follow the proof of [27, 5.1.3]. This proof uses [27, 5.1.5] to consider maps of the form $L_i A \rightarrow L_i B$. The key point for the analogous statement here is to use the right adjoint G_i of the functor L_i (instead of

the constant functor.) For an object X in $\mathbb{M}(i)$, the \mathbb{M} diagram $G_i X$ at $a \in \mathbf{D}_i$ is $G_a X$. Since each G_a is a right Quillen functor, G_i takes (trivial) fibrations to objectwise (trivial) fibrations just as was required of the constant functor.

The only other change needed in the proof of [27, 5.1.3] is that for the induction step in the construction of the functorial factorizations one uses factorization in $\mathbb{M}(i)$ to factor the map $X(i) \coprod_{L_i X} L_i Y \rightarrow Y(i)$. \square

Remark B.2. In [30] Reedy diagrams are considered where the model structure is allowed to vary although the underlying category does not vary. Johnson's results point to the possibility that one could further generalize the theorem above to consider Reedy diagram categories.

In this paper, we only consider diagram categories \mathbf{D} with at most one map between any two objects. Restricting to this situation simplifies the arguments for the following proposition.

Proposition B.3. *Let \mathbf{D} be a direct (or inverse) category with at most one map between any two objects. Consider, as above, a diagram \mathbb{M} over \mathbf{D} of proper and cellular model categories. Then the two model structures of diagrams over \mathbb{M} defined in Theorem B.1 are proper, cellular model categories.*

Proof: We first establish properness. In the diagram-projective case fibrations and weak equivalences are defined objectwise and one can show that any cofibration induces an objectwise cofibration. Since pullbacks and pushouts are constructed at each object and $\mathbb{M}(i)$ is assumed to be a proper model structure for each i , properness follows. The diagram-injective case is dual.

Next we show these model structures are cellular. We use Hirschhorn's treatment of Reedy categories here [26, Chapter 15]. Note that a direct category is an example of a Reedy category with no morphisms that lower degrees. Here the matching categories are empty so that the matching objects are just the terminal object. Thus the Reedy fibrations are just the objectwise fibrations and the Reedy model structure [26, 15.3.4] agrees with the diagram-projective model structure defined above. The arguments for an inverse category are dual.

Next we define the generating (trivial) cofibrations. Given an object X in $\mathbb{M}(i)$, define the free \mathbb{M} diagram generated by X at i to be $\mathcal{F}_X^i(j) = F_a X$ when $\mathbf{D}(i, j) = \{a\}$ is non-empty and the initial object otherwise. In [26, 15.6.18], the boundary $\partial \mathcal{F}^i$ of a free functor is defined for general Reedy categories and uses a non-identity map in \mathbf{D} which lowers degree. This definition simplifies here; for \mathbf{D} a direct category, $\partial \mathcal{F}^i$ is the diagram of initial objects since no map lowers degree and for \mathbf{D} an inverse category $\partial \mathcal{F}_X^i(j) = F_a X$ for $\mathbf{D}(i, j) = \{a\}$ with $i \neq j$ and is the initial object otherwise. Given a map $f : A \rightarrow B$ in $\mathbb{M}(i)$, let RF_f^i denote the \mathbb{M} diagram map

$$\mathcal{F}_A^i \coprod_{\partial \mathcal{F}_A^i} \partial \mathcal{F}_B^i \rightarrow \mathcal{F}_B^i.$$

Note, for \mathbf{D} a direct category, RF_f^i is just $\mathcal{F}_A^i \rightarrow \mathcal{F}_B^i$. Let I_i denote the generating cofibrations for $\mathbb{M}(i)$. Let $RF_f^{\mathbf{D}}$ denote the set of maps RF_f^i for all maps f in I_i for all i in \mathbf{D} . Define $RF_f^{\mathbf{D}}$ similarly based on the sets J_i of generating trivial cofibrations for $\mathbb{M}(i)$. By [26, 15.6.27],

both the diagram-projective and the diagram-injective model structure on \mathbb{M} diagrams are cofibrantly generated with generating cofibrations $RF_I^{\mathbf{D}}$ and generating trivial cofibrations $RF_J^{\mathbf{D}}$.

Finally, [26, 15.7.6] establishes the additional conditions for showing this is a cellular model category given that each category $\mathbb{M}(i)$ is a cellular model category. \square

We end this section with an application of the Cellularization Principle, Proposition A.6, which is a general model for many of its applications in this paper. Assume given an inverse category \mathbf{D} with at most one morphism in each $\mathbf{D}(i, j)$ and a diagram of ring spectra, R , indexed on \mathbf{D} . We consider the associated diagram of model categories \mathbb{M} with $\mathbb{M}(i)$ the model category of $R(i)$ -module spectra and $F_a = R(j) \wedge_{R(i)} (-)$ the left Quillen functor given by extension of scalars. We refer to \mathbb{M} -diagrams as R -modules. Here, we compare the diagram-injective model category of R -modules with modules over the homotopy inverse limit of the diagram R . By [26, 19.9.1], the homotopy inverse limit of R is the inverse limit of a fibrant replacement of R in the diagram-injective model category of \mathbf{D} -diagrams of ring spectra. This model structure exists by [27, 5.1.3]; see also [26, 15.3.4] since an inverse category is a particular example of a Reedy category. Let $g : R \rightarrow fR$ be this fibrant replacement and let \overline{R} denote the inverse limit over \mathbf{D} of fR . We then compare R -modules and \overline{R} -modules via the category of fR -modules. Since $R \rightarrow fR$ is an objectwise weak equivalence, there is a Quillen equivalence between R -modules and fR -modules by Lemma B.5 below. We also establish below a Quillen adjunction between \overline{R} -modules and fR -modules which is a Quillen equivalence after cellularization. This leads to the following statement.

Proposition B.4. *There is a zig-zag of Quillen equivalences between the category of \overline{R} -modules and the cellularization with respect to R of R -modules.*

$$\overline{R}\text{-mod} \simeq_Q R\text{-cell-}R\text{-mod}$$

We first need the following lemma.

Lemma B.5. *Assume given $L : \mathbb{M} \rightarrow \mathbb{N}$ a map of diagrams of model categories over a direct category \mathbf{D} . If each $L(i) : \mathbb{M}(i) \rightarrow \mathbb{N}(i)$ is a left Quillen equivalence, then L induces a Quillen equivalence between the diagram-projective model structures between the associated categories of \mathbb{M} and \mathbb{N} diagrams. If \mathbf{D} is an inverse category, the corresponding statement holds for diagram-injective model structures.*

Proof: This follows since a cofibrant or fibrant \mathbb{M} diagram is objectwise cofibrant or fibrant in either the diagram-projective or diagram-injective model structure. Namely, given a cofibrant \mathbb{M} diagram X and a fibrant \mathbb{N} diagram Y , a map $LX \rightarrow Y$ is an objectwise weak equivalence if and only if $X \rightarrow RY$ is an objectwise weak equivalence since L and its right adjoint R are objectwise Quillen equivalences. \square

Proof of Proposition B.4: Since $g : R \rightarrow fR$ is an objectwise weak equivalence, and extension of scalars along weak equivalences of ring spectra induce Quillen equivalences, the associated diagram module categories are Quillen equivalent by Lemma B.5. By Corollary A.7, this induces a Quillen equivalence on the cellularizations of the diagram-injective

model structures

$$R\text{-cell-}R\text{-mod} \simeq_Q fR\text{-cell-}fR\text{-mod}$$

since R is cofibrant in R -modules and extension of scalars takes R to fR .

Next we compare \overline{R} -modules and fR -modules. Since \overline{R} is the inverse limit of fR , any fR -module M defines an underlying \mathbf{D} diagram of \overline{R} -modules \widetilde{M} . Denote the inverse limit of \widetilde{M} by \overline{M} . The functor $fR \otimes_{\overline{R}} -$ is left adjoint to this inverse limit functor and takes an \overline{R} module N to the fR -module with $fR(i) \otimes_{\overline{R}} N$ at $i \in \mathbf{D}$. Since extension of scalars for a map of ring spectra is a left Quillen functor and cofibrations and weak equivalences are defined objectwise, $fR \otimes_{\overline{R}} -$ is a left Quillen functor. We next apply the Cellularization Principle, Proposition A.6 (2), to this Quillen adjunction to induce a Quillen equivalence on the appropriate cellularizations. Note that \overline{R} is cofibrant as an \overline{R} -module and applying extension of scalars to it gives fR . Since fR is diagram-injective fibrant as a diagram of ring spectra, it is also diagram-injective fibrant as an fR -module. Since \overline{R} is the inverse limit of fR , cellularization induces a Quillen equivalence.

$$\overline{R}\text{-cell-}\overline{R}\text{-mod} \simeq_Q fR\text{-cell-}fR\text{-mod}$$

Since \overline{R} is already a cofibrant generator of \overline{R} -modules, the cellular weak equivalences and fibrations in $\overline{R}\text{-cell-}\overline{R}\text{-mod}$ agree with those before cellularization. Thus the cellularization of the model structure on the left is unnecessary and the statement follows. \square

Remark B.6. We want to point out that when \mathbf{D} is the indexing category for pullback shaped diagrams, the model category $R\text{-cell-}R\text{-mod}$ is similar to the homotopy fibre product homotopy theory considered in [6] and [48]. In this particular case though we can identify this more simply as \overline{R} -modules.

Remark B.7. Here we consider an inverse category \mathbf{D} with at most one morphism in each $\mathbf{D}(i, j)$. In this remark, we note that the categories of module spectra over diagrams of ring spectra over \mathbf{D} considered in the end of this section are equivalent to categories of module spectra over spectral categories (ring spectra with many objects.) If $\mathbf{D}(i, j)$ is non-empty then the map $R(i) \rightarrow R(j)$ makes $R(j)$ an $R(i)$ -module. There is an associated spectral category indexed on the objects of \mathbf{D} which we also denote by R with $R(i, i)$ the ring $R(i)$, $R(i, j)$ trivial when $\mathbf{D}(i, j)$ is empty, and $R(i, j)$ the $R(j) - R(i)$ bimodule $R(j)$ when $\mathbf{D}(i, j)$ is non-empty.

A (left) module M over R is a covariant spectrally enriched functor from R to Spectra. The data needed to specify such a module is exactly the same as given for a module over the associated diagram R . First, for each object i in \mathbf{D} , $M(i)$ is an $R(i, i)(= R(i))$ module spectrum and for each morphism $a : i \rightarrow j$ in \mathbf{D} the module structure specifies a map $R(i, j) \otimes_{R(i, i)} M(i) \rightarrow M(j)$. Since $R(i, j) = R(j)$, this is the required map $F_a M(i) \rightarrow M(j)$ where F_a is extension of scalars over $R(i) \rightarrow R(j)$.

We consider covariant functors here because this eases the comparison with diagrams even though this differs from the right modules (or contravariant functors) considered in [43, 3.3.2].

APPENDIX C. DOUBLY INJECTIVE MODEL STRUCTURES

In this section we develop a model structure on the category of modules over a DGA with many objects, $R\text{-mod}$, with weak equivalences determined objectwise and cofibrations the monomorphisms. This coincides with the doubly injective, or diagram-injective and algebraically injective, model structure discussed in Section 16.B.

Proposition C.1. *The doubly injective model structure on $R\text{-mod}$, with objectwise weak equivalences and cofibrations the monomorphisms is a combinatorial Quillen model category.*

This is a special case of the more general result in [25] which shows that combinatorial model structures satisfying certain mild hypotheses have Quillen equivalent model structures with cofibrations the monomorphisms (and the same weak equivalences). Here we start with the standard model structure on $R\text{-mod}$ with objectwise weak equivalences and fibrations from [44, 6.1] and [46, 3.1].

To prove this proposition we use Smith’s argument for constructing combinatorial model categories; this first appeared in [5, 1.7], but is also similar to [34, A.2.6.8] and [40, 4.3]. A model structure is *combinatorial* if it is cofibrantly generated and the underlying category is locally presentable. The underlying category of $R\text{-mod}$ is locally presentable; this follows by [5, 3.10], for example, since this category is a Grothendieck abelian category and has a generator (the coproduct of the free modules at each object). The standard model structure on $R\text{-mod}$ from [44, 6.1] is then combinatorial since it is cofibrantly generated.

Proof of Proposition C.1: To use [5, 1.7] to establish the doubly injective model structure, we must first establish a set of morphisms I which generates the monomorphisms. Such a set I exists by [5, 1.12] since $R\text{-mod}$ is locally presentable, monomorphisms here are closed under transfinite compositions, and subobjects have effective unions.

We now verify the criteria listed in [5, 1.7] for W the object-wise quasi-isomorphisms and I this set which generates the monomorphisms. Since W is the class of weak equivalences in the standard model category structure on $R\text{-mod}$, it follows that W is closed under retracts, has the 2-out-of-3 property, and is accessible by [34, A.2.6.6] or [40, 4.1] and hence satisfies the solution set condition [5, 1.15]. We next consider the class of maps $\text{inj}(I)$, those maps with the right lifting property with respect to the set I . This class agrees with the class with the right lifting property with respect to all monomorphisms by [5, 1.3]. Since the cofibrations in the standard model structure are monomorphisms, any map in $\text{inj}(I)$ will be a weak equivalence (in fact, a trivial fibration in the standard model structure.) Finally, one can check that the class of weak equivalences which are also monomorphisms is closed under transfinite composition and pushouts. \square

REFERENCES

- [1] V. Angeltveit Enriched Reedy categories Proc. Amer. Math. Soc. **136** (2008), no. 7, pp. 2323-2332.
- [2] A. K. Bousfield and V. K. A. M. Gugenheim *On PL de Rham theory and rational homotopy type* Mem. Amer. Math. Soc. **8** (1976), no. 179, ix+94 pp.
- [3] D.Barnes “Rational Equivariant Spectra.” Thesis (2008), University of Sheffield, arXiv:0802.0954
- [4] D.Barnes “Classifying Dihedral $O(2)$ -Equivariant Spectra.” Preprint (2008) arXiv:0804.3357
- [5] T. Beke “Sheafifiable homotopy model categories.” Math. Proc. Camb. Phil. Soc. **129** (2000), no.3, pp.447-475

- [6] Julia Bergner *Homotopy fiber products of homotopy theories*, to appear in Israel J. Math, arXiv:0811.3175v3
- [7] D. Dugger *Spectral enrichments of model categories* Homology, Homotopy and Applications **8** (2006), No. 1, 1-30.
- [8] W.G.Dwyer and J.P.C.Greenlees "The equivalence of categories of torsion and complete modules." American Journal of Mathematics **124** (2002) 199-220.
- [9] W.G.Dwyer, J.P.C.Greenlees and S.B.Iyengar "Duality in algebra and topology." Advances in Maths **200** (2006) 357-402
- [10] W. G. Dwyer and J. Spaliński Homotopy theories and model categories. *Handbook of algebraic topology*, 73–126, North-Holland, Amsterdam, 1995.
- [11] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules and Algebras in Stable Homotopy Theory*, Volume 47 of *Amer. Math. Soc. Surveys and Monographs*. American Mathematical Society, 1996.
- [12] J.P.C.Greenlees "A rational splitting theorem for the universal space for almost free actions." Bull. London Math. Soc. **28** (1996) 183-189.
- [13] J.P.C.Greenlees "Rational Mackey functors for compact Lie groups." Proc. London Math. Soc **76** (1998) 549-578
- [14] J.P.C.Greenlees "Rational $O(2)$ -equivariant cohomology theories." Fields Institute Communications **19** (1998) 103-110
- [15] J.P.C.Greenlees "Rational S^1 -equivariant stable homotopy theory." Mem. American Math. Soc. **661** (1999) xii + 289 pp
- [16] J.P.C.Greenlees "Rational $SO(3)$ -equivariant cohomology theories." Contemporary Maths. **271**, American Math. Soc. (2001) 99-125
- [17] J.P.C.Greenlees "Triangulated categories of rational equivariant cohomology theories." Oberwolfach Reports 8/2006, 480-488
- [18] J.P.C.Greenlees "Rational torus-equivariant cohomology theories I: calculating groups of stable maps." JPAA **212** (2008) 72-98
- [19] J.P.C.Greenlees "Rational torus-equivariant cohomology theories II: the algebra of localization and inflation." Preprint (Second edition, 2009) 23pp
- [20] J.P.C.Greenlees "Rational S^1 -equivariant elliptic cohomology." Topology **44**(2005) 1213-1279
- [21] J.P.C.Greenlees "Algebraic groups and equivariant cohomology theories." Proceedings of 2002 Newton Institute workshop 'Elliptic cohomology and chromatic phenomena', Cambridge University Press, (2007) 89-110pp
- [22] J.P.C.Greenlees and J.P.May "Generalized Tate cohomology" Mem. American Math. Soc. **543** (1995) 178pp
- [23] J.P.C.Greenlees and B.E.Shipley "An algebraic model for free rational G -spectra for connected compact Lie groups G ." Math. Z. (2011) 29pp DOI 10.1007/s00209-010-0741-2
- [24] J.P.C.Greenlees and B.E.Shipley "An algebraic model for free rational G -spectra." Preprint (2011) 8pp arXiv 1101.4818
- [25] K. Hess and B. Shipley *The homotopy theory of coalgebras over a comonad*, in preparation.
- [26] P. S. Hirschhorn *Model categories and their localizations* Mathematical Surveys and Monographs, **99**, American Mathematical Society, Providence, RI, 2003. xvi+457 pp.
- [27] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, **63**, American Mathematical Society, Providence, RI, 1999, xii+209 pp.
- [28] M. Hovey, *Spectra and symmetric spectra in general model categories* J. Pure Appl. Algebra **165** (2001), no. 1, 63–127.
- [29] M. Hovey, B. Shipley and J. Smith, *Symmetric spectra* J. Amer. Math. Soc. **13** (2000), 149-209.
- [30] M. W. Johnson *On modified Reedy and modified projective model structures* Theory and Applications of Categories, **24**, 2010, No. 8, pp 179-208.
- [31] G. M. Kelly, *Basic concepts of enriched category theory*, Cambridge Univ. Press, Cambridge, 1982, 245 pp.
- [32] I. Kriz and J. P. May, *Operads, algebras, modules and motives* Astérisque **233** (1995), iv+145pp.
- [33] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure), *Equivariant stable homotopy theory* Lecture Notes in Mathematics **1213**, Springer-Verlag, 1986.

- [34] J. Lurie, *Higher topos theory*. Annals of Mathematics Studies, 170. Princeton University Press, 2009. arXiv:math/0608040.
- [35] J.E.McClure “ E_∞ -ring structures for Tate spectra.” Proc. Amer. Math. Soc. 124 (1996), no. 6, 1917-1922.
- [36] M. Mandell and J. P. May, *Equivariant Orthogonal Spectra and S-Modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108 pp.
- [37] M. Mandell, J. P. May, S. Schwede and B. Shipley *Model categories of diagram spectra* Proc. London Math. Soc. **82** (2001), 441-512.
- [38] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, **43**, Springer-Verlag, 1967.
- [39] D.G.Quillen, “Rational homotopy theory.” Ann. of Math. **90** (1969) 205-295.
- [40] J. Rosicky *On combinatorial model categories*. Appl. Cat. Str. 17 (2009), 303-316.
- [41] S. Schwede “S-modules and symmetric spectra.” Math. Ann. 319 (2001), no. 3, 517-532.
- [42] S. Schwede and B. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. **80** (2000), 491-511.
- [43] S. Schwede and B. Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), no. 1, 103-153.
- [44] S. Schwede and B. Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. **3** (2003), 287-334.
- [45] B. Shipley *An algebraic model for rational S^1 -equivariant stable homotopy theory*, Quart. J. Math. **53** (2002), 87-110.
- [46] B. Shipley *HZ-algebra spectra are differential graded algebras*, Amer. J. of Math. 129 (2007) 351379.
- [47] B. Shipley *A convenient model category for commutative ring spectra*, Contemp. Math. **346** (2004), 473-483.
- [48] Bertrand Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. **167** (2007), no. 3, 615-667.

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